

## Determination of dynamic properties of frames with viscoelastic dampers

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### Abstract

In this note, frame structures with viscoelastic dampers mounted on them are considered. The viscoelastic dampers (VE dampers) are modeled using the fractional rheological model. The structures are treated as linear elastic systems. The equation of motion of the whole system (structure with dampers) is written in terms of state-space variables. A nonlinear eigenvalue problem is formulated from which dynamic parameters of the system can be determined. The Newton method is used to solve the nonlinear eigenvalue problem. Results of the calculation will be also presented and briefly discussed.

*Keywords: viscoelastic dampers, rheological models, frequency, damping ratio, nonlinear eigenvalue problem*

### 1. Introduction

In civil engineering viscoelastic dampers (VE dampers) are successfully used to reduce vibrations caused by wind and earthquake. It was found that incorporation of the VE dampers in a structure leads to significant reduction of unwanted vibrations [1]. In the past, several rheological models were proposed to describe the dynamic behaviour of VE materials and dampers. Both the classical and the fractional models of dampers are available. Descriptions of these models are given in [2, 3].

In this note, frame structures with the viscoelastic dampers mounted on it are considered. The VE dampers are modeled using the fractional rheological model. The structures are treated as linear elastic systems. The equation of motion of the whole system (the structure with the dampers) is written in terms of state-space variables. The resulting matrix equation of motion is a fractional differential equation.

The paper is devoted to determination of dynamic parameters of considered structures. The nonlinear eigenvalue problem is formulated from which the dynamic parameters of the system can be determined. The Newton method is used to solve the above mentioned nonlinear eigenvalue problem.

Results of the calculation will be also presented and briefly discussed. The influence of the key parameter, which describes the order of fractional derivative, on the dynamic parameters of a frame with VE dampers is shown.

### 2. Damping model of viscoelastic dampers

The dynamic behaviour of a damper „*r*” may be written in the form of a differential equation, appropriate for rheological models: Maxwell, Kelvin-Voigt and standard one:

$$\begin{aligned} c_{di}(D^\alpha q_{si} - D^\alpha q_{di}) - k_{di}q_{di} &= 0, \\ u_i &= k_{di}q_i + c_{di}D^\alpha q_i, \\ u_i + \tau_{di}D^\alpha u_i &= c_{di}D^\alpha q_i + k_{di}q_i. \end{aligned} \quad (1)$$

Equation (1) expresses a relation between the force  $u_i$  and the relative nodal displacement  $q_i$ . The symbol  $D^\alpha f$  denotes the fractional derivative of order  $\alpha$ ,  $\tau_{di} = c_{di}/k_{di}$  is the relaxation time,  $c_{di}$  and  $k_{di}$  are damping and elastic parameters of the damper, respectively. In this study we apply the definition of the fractional order derivative proposed by the Riemann-

Liouville. The relation (1.1) for Maxwell model was derived using an internal variable  $q_{di}$  (see Fig. 1).

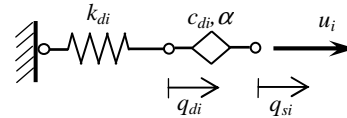


Figure 1: The diagram of fractional Maxwell model

### 3. State-space formulation for a frame with VE dampers

The equation of motion for a structure with viscoelastic dampers mounted on it may be derived assuming the fractional Maxwell model for the damper. In the case of a multistorey frame with a number of dampers, the equation of motion and expression (1) can be written as:

$$\mathbf{E}_1 \mathbf{C}_d D^\alpha \mathbf{q}_d - \mathbf{E}_1 \mathbf{C}_d \mathbf{E}_1^T D^\alpha \mathbf{q}_s - \mathbf{C}_s \dot{\mathbf{q}}_s - \mathbf{M}_s \ddot{\mathbf{q}}_s + \mathbf{E}_2 \mathbf{K}_d \mathbf{q}_d - (\mathbf{K}_s + \mathbf{E}_2 \mathbf{K}_d \mathbf{E}_2^T) \mathbf{q}_s = \mathbf{P}(t), \quad (2)$$

$$-\mathbf{C}_d D^\alpha \mathbf{q}_d + \mathbf{C}_d \mathbf{E}_1^T D^\alpha \mathbf{q}_s - \mathbf{K}_d \mathbf{q}_d + \mathbf{K}_d \mathbf{E}_2^T \mathbf{q}_s = \mathbf{0}, \quad (3)$$

where:  $\mathbf{q}_d$  is the vector of internal variables,  $\mathbf{q}_s$  is the vector of structure displacements,  $\mathbf{C}_s$ ,  $\mathbf{M}_s$ ,  $\mathbf{K}_s$  are damping, mass and stiffness matrices of the structure, respectively,  $\mathbf{C}_d$  and  $\mathbf{K}_d$  represent matrices of damping and elastic parameters of the dampers, respectively,  $\mathbf{E}_1$  and  $\mathbf{E}_2$  are so-called allocation matrices,  $\mathbf{P}(t)$  is the vector of external forces.

Letting  $\mathbf{P}(t) = \mathbf{0}$ , introducing an additional equation

$$-\mathbf{M}_s \dot{\mathbf{q}}_s + \mathbf{M}_s \dot{\mathbf{q}}_s = \mathbf{0}, \quad (4)$$

and the vector of state variables  $\mathbf{z} = \text{col}(\mathbf{q}_d, \mathbf{q}_s, \dot{\mathbf{q}}_s)$ , one may rewrite the expressions (2) – (4) in the following form:

$$\mathbf{A}_1 D^\alpha \mathbf{z} + \mathbf{A}_2 \dot{\mathbf{z}} + \mathbf{B} \mathbf{z} = \mathbf{0}, \quad (5)$$

where:

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{C}_d & \mathbf{C}_d \mathbf{E}_1^T & \mathbf{0} \\ \mathbf{E}_1 \mathbf{C}_d & -\mathbf{E}_1 \mathbf{C}_d \mathbf{E}_1^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C}_s & -\mathbf{M}_s \\ \mathbf{0} & -\mathbf{M}_s & \mathbf{0} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} -\mathbf{K}_d & \mathbf{K}_d \mathbf{E}_2^T & \mathbf{0} \\ \mathbf{E}_2 \mathbf{K}_d & -\mathbf{K}_s - \mathbf{E}_2 \mathbf{K}_d \mathbf{E}_2^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_s \end{bmatrix}. \quad (6)$$

**4. Dynamic properties of a frame structure with VE dampers mounted on it**

Applying the Laplace transform, and taking into account:

$$\mathcal{L}[\mathbf{z}(t)] = \mathbf{Z}(s), \quad \mathcal{L}[D^\alpha \mathbf{z}(t)] = s^\alpha \mathbf{Z}(s), \quad \mathcal{L}[\dot{\mathbf{z}}(t)] = s \mathbf{Z}(s), \quad (7)$$

the equation of motion (5) can be written as:

$$(s^\alpha \mathbf{A}_1 + s \mathbf{A}_2 + \mathbf{B}) \mathbf{Z} = \mathbf{0}. \quad (8)$$

Note that the relation (8) describes the nonlinear eigenproblem, which can be solved using the continuation method with a parameter  $\alpha$ .

In the first step, after introducing  $\alpha = 1$  into Eqn. (8), one obtains the following linear eigenproblem:

$$[\mathbf{B} + s(\mathbf{A}_1 + \mathbf{A}_2)] \mathbf{Z} = \mathbf{0}. \quad (9)$$

The solution of Eqn. (9) leads to the eigenvalues  $s_k$  and eigenvectors  $\mathbf{z}_k$ . These results give us one point (for  $\alpha = 1$ ) at curves  $s_k(\alpha)$  and  $\mathbf{z}_k(\alpha)$ .

Then for a chosen pair of variables  $s_k(\alpha)$  and  $\mathbf{z}_k(\alpha)$  the next points on the considered curves are computed. We introduce the additional condition associated with Eqn. (8)

$$f_2 = \frac{1}{2} \mathbf{z}_k^T \mathbf{h} - a = \frac{1}{2} \mathbf{z}_k^T (\alpha s^{\alpha-1} \mathbf{A}_1 + \mathbf{A}_2) \mathbf{z}_k - a = 0, \quad (10)$$

where  $a$  has a given value. The condition (10) may be treated as a way of normalization of the eigenvector  $\mathbf{z}_k$ .

Let us assume, that we investigate the solution of the system (8) and (10) for  $\alpha_j = \alpha_{j-1} + \Delta\alpha$ , where  $j$  denotes the increment number. Moreover, we know the approximate solution denoted as  $s_k^{(i)}(\alpha_j)$  and  $\mathbf{z}_k^{(i)}(\alpha_j)$  obtained at the iteration step  $(i-1)$ . The incremental equations of the Newton method, associated with conditions (8) and (10), have the following form:

$$\mathbf{G} d\mathbf{z}_k + \mathbf{h} ds_k = -\mathbf{f}_1, \quad \mathbf{h}^T d\mathbf{z}_k + g ds_k = -f_2, \quad (11)$$

where:

$$g = \frac{1}{2} \alpha(\alpha-1) s_k^{\alpha-2} \mathbf{z}_k^T \mathbf{A}_1 \mathbf{z}_k, \quad \mathbf{f}_1 = (s_k^\alpha \mathbf{A}_1 + s_k \mathbf{A}_2 + \mathbf{B}) \mathbf{z}_{k-1}.$$

$$\mathbf{G} = s_k^\alpha \mathbf{A}_1 + s_k \mathbf{A}_2 + \mathbf{B}, \quad \mathbf{h} = (\alpha s_k^{\alpha-1} \mathbf{A}_1 + \mathbf{A}_2) \mathbf{z}_k. \quad (12)$$

To obtain the next approximation we use the formulas:

$$s_k^{(i+1)}(\alpha_j) = s_k^{(i)}(\alpha_j) + ds_k, \quad \mathbf{z}_k^{(i+1)}(\alpha_j) = \mathbf{z}_k^{(i)}(\alpha_j) + d\mathbf{z}_k. \quad (13)$$

where  $\varepsilon_1$  and  $\varepsilon_2$  represent an assumed accuracy of the calculations. One may finish the iteration process when:

$$|ds_k| \leq \varepsilon_1 |s_k^{(i+1)}(\alpha_j)|, \quad \|d\mathbf{z}_k\| \leq \varepsilon_2 \|\mathbf{z}_k^{(i+1)}(\alpha_j)\|. \quad (14)$$

**5. Numerical results**

Let us consider a two-storey frame structure (masses of floors 216 and 172,8 Mg, column height 3,0 m and bending rigidity 11685 kNm<sup>2</sup>, beam span 6,0 m and bending rigidity 47416 kNm<sup>2</sup>), with a viscoelastic damper mounted on the second storey. The dynamic behaviour of the damper was

defined by the classical and fractional Maxwell, Kelvin and standard models. Moreover, we assume  $k_d = k_{11}$ , where  $k_{11}$  is the element of the matrix  $\mathbf{K}_s$ , and various values of damping coefficient  $c_d$ , such that  $\tau_d \in (0.01, 0.20)$ , which refer to commonly applied dampers. At Figure 2 the plots of nondimensional damping ratio  $\gamma$  are presented in relation to the value of the damping parameter  $c$  for the considered rheological models: Maxwell, Kelvin, standard and fractional Maxwell one ( $M$ ,  $K$ ,  $S$  and  $fM$ , respectively). The above mentioned ratio  $\gamma$  and the natural frequency  $\omega$  are defined as:

$$\omega_i^2 = \mu_i^2 + \eta_i^2, \quad \gamma_i = -\mu_i / \omega_i, \quad \mu_i = \text{Re}(s_i), \quad \eta_i = \text{Im}(s_i). \quad (15)$$

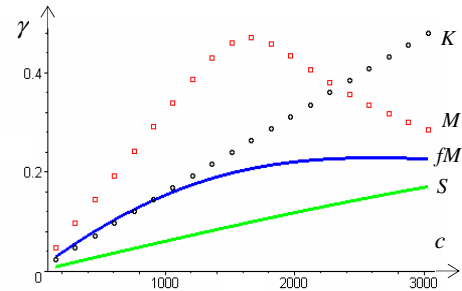


Figure 2: The nondimensional damping ratio  $\gamma$  versus the damping parameter  $c_d$

Results of the numerical calculations carried out for the frame with the damper described by the fractional Maxwell model are presented at Fig. 3. There is shown the influence of the parameter  $\alpha$  on the nondimensional damping ratio  $\gamma$ .

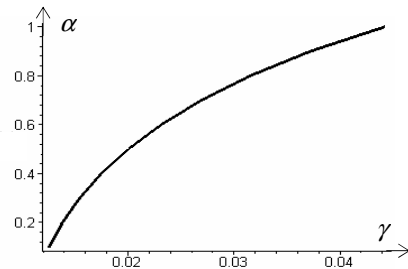


Figure 3: The nondimensional damping ratio  $\gamma$  versus the parameter  $\alpha$

**6. Conclusions**

This paper presents a methodology to calculate dynamic parameters of structures with viscoelastic dampers mounted on it. The dynamic behaviour of the dampers is described by classical and fractional rheological models. A nonlinear eigenproblem was solved applying the continuation method.

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