

Remarks on amplitude equation and Rayleigh quotient for free vibration of structures with cubic non-linearity

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ABSTRACT: In the paper, several aspects concerning the formulation and solution of amplitude equations for free vibration of systems with cubic non-linearity is discussed in details. The well-known concept of the Rayleigh quotient is extended to a non-linear case of interest. Moreover, the introduced Rayleigh quotient is used to develop a new and efficient method to solve amplitude equations. The accuracy of solutions obtained by means of some widely used methods of analysis of free vibration problems is also analysed. It was found that the Galerkin method gives us most accurate results.

1. INTRODUCTION

An analysis of the problem of free vibration of non-linear oscillating systems leads to determination of frequencies and modes of vibration. Since these quantities are the essentials characteristics of each vibrating system, a lot of attention is given to the problem of free vibration. In the case of linear vibrating systems, the exact solution of motion equation is known and the problem of determination of frequencies and modes of vibration is reduces to finding a solution of the eigenvalue problem. The eigenvalue problem could be solved using many methods, like the subspace iteration method, the Lanczos method, the vector iteration methods and other ones. In these methods the Rayleigh quotient is often used. A description of mentioned methods could be found in many monographs (see, [1,2]).

The problems of free vibration of non-linear systems are much more complex. Beside a few exceptions, the exact solutions of motion equations are not known and only approximate solutions are available. Very often in the case of non-linear continuous systems, the approximate solutions of motion equations are written as a sum of products of time and space functions. To give an example, for beams these solutions can be written in a form:

$$w(x, t) = v_i(x)q_i(t) , \quad (1)$$

where $v_i(x)$ and $q_i(t)$ are the functions of space x and time t , respectively. Moreover, $i=1,2,\dots,n$ and the summation convention for repeated indices is used in (1) and throughout this work. Usually, the approximate solution in the time domain is assumed in a form of the truncated Fourier series and then $q_i(t) = \cos i \boldsymbol{\omega} t$, where $\boldsymbol{\omega}$ = the non-linear frequency of vibration.

Additionally, in many instances only the fundamental harmonic of Fourier series is taken into account, i.e. $n=1$ in (1). The space functions $v_i(x)$ are usually treated as unknown quantities or it is assumed they are a linear combination of trial functions. In a later case

$$v_i(x) = \mathbf{a}_{ij} u_j(x) , \quad (2)$$

where $j=1,2,\dots,m$, $u_j(x)$ = the trial function and \mathbf{a}_{ij} = the unknown factor. Quite frequently, the modes of vibration of linearised system are chosen as the trial functions. A similar description is used in the finite element method and then the relation (2) refers to the typical finite element. Moreover, in many cases it is assumed that $m=1$.

Using the harmonic balance method, the Galerkin method or the Ritz method, it is possible to reduce the considered problem to finding a solution for a set of non-linear, ordinary differential equations (if the unknown quantities are $v_i(x)$ and \boldsymbol{w}) or for a system of non-linear algebraic equations (if \mathbf{a}_{ij} and \boldsymbol{w} are unknowns). The above-mentioned system of non-linear algebraic equations will be referred to as the amplitude equations. From a mathematical point of view these equations can be considered as the non-linear eigenvalue problem.

In this paper, the analysis of second approach reducing the problem to the amplitude equations is of particular interest. Our considerations are restricted to non-linear dynamical systems with a cubic characteristic of restoring forces.

2. DERIVATION OF AMPLITUDE EQUATIONS

Strings, beams, membranes and plates are examples of structures for which the geometrically non-linear theory

must be used to correctly describe responses of such structures under sufficiently large external forces. Moreover, the internal forces of these structures can be written as cubic functions of displacements.

In a case of free vibration, the motion equations of undamped structures with cubic characteristic, treated as the discrete systems, can be written in the following matrix form:

$$\mathbf{M}\ddot{\mathbf{w}}(t) + \left[\mathbf{K}_0 + \frac{1}{2}\mathbf{K}_2(\mathbf{w}(t))\right]\mathbf{w}(t) = \mathbf{0}, \quad (3)$$

where \mathbf{M} = the global mass matrix, \mathbf{K}_0 = the global, linear stiffness matrix and $\mathbf{K}_2(\mathbf{w}(t))$ = the global, non-linear stiffness matrix, $\mathbf{w}(t)$ = the global vector of nodal parameters and dots means differentiation with respect to t . It is useful to note that, on a finite element level, the non-linear stiffness matrix $\mathbf{K}_2^e(\mathbf{w}_e(t))$ can be written in the form (see [3,4]):

$$\mathbf{K}_2^e(\mathbf{w}_e) = \int_V \mathbf{B}^T(\mathbf{w}_e) \mathbf{E} \mathbf{B}(\mathbf{w}_e) dV, \quad (4)$$

where \mathbf{E} = the matrix of elasticity and V = the finite element volume. The matrix $\mathbf{B}(\mathbf{w})$ is the linear and homogenous function of nodal parameters. It means that, the non-linear stiffness matrix is a quadratic and homogenous function of nodal parameters. It is assumed that the finite element method is used to discretize the continuous systems. The large displacements and small rotations were assumed during the process of derivation of the motion equation (3). Derivation of Equation (3) is given in detail in [3]. The kinetic energy K and the strain energy U of the structure can be written in the following form (see [3]):

$$K = \frac{1}{2} \dot{\mathbf{w}}^T(t) \mathbf{M} \dot{\mathbf{w}}(t), \quad (5)$$

$$U = \frac{1}{2} \mathbf{w}^T(t) \mathbf{K}_0 \mathbf{w}(t) + \frac{1}{8} \mathbf{w}^T(t) \mathbf{K}_2(\mathbf{w}(t)) \mathbf{w}(t). \quad (6)$$

In this paper, the beam structures are considered as example structures. For beams (see [3,5])

$$\mathbf{K}_0 = \mathbf{K}, \quad \mathbf{K}_2(\mathbf{w}(t)) = EA/l \mathbf{B} \mathbf{w}(t) \mathbf{w}^T(t) \mathbf{B}, \quad (7)$$

where E = the Young's constant, A = the area of of beam cross-section, l = the beam length and \mathbf{B} = the matrix of geometric stiffness.

The simplest method for derivation of the amplitude equation is proposed by Mei [6]. Subsequently, this method was used in many papers devoted to non-linear dynamic problems of beams and plates [7 - 11]. This method assumes that the solution of motion equation fulfils the conditions:

$$\ddot{\mathbf{w}}_{\max}(t_1) = -\mathbf{w}^2 \mathbf{w}_{\max}(t_1), \quad \dot{\mathbf{w}}(t_1) = \mathbf{0}, \quad (8)$$

at the point of the reversal of motion (i.e. the point of maximum amplitude). Conditions (8) can be also understood as the collocation conditions.

The above conditions are fulfilled if only the fundamental harmonic is taken into account in the approximate solution of motion equation i.e.

$$\mathbf{w}(t) = \mathbf{a} \cos \mathbf{w}. \quad (9)$$

Introducing (9) into the equation of motion (3) and using conditions (8) the matrix amplitude equation is obtained in the form:

$$\left(\mathbf{K}_0 - \mathbf{w}^2 \mathbf{M} + \frac{1}{2} \mathbf{K}_2(\mathbf{a}, \mathbf{a})\right) \mathbf{a} = \mathbf{0}. \quad (10)$$

The notation like $\mathbf{K}_2(\mathbf{a}, \mathbf{a})$ underlines that this matrix is the quadratic function of amplitudes of nodal parameters.

Very popular methods of derivation of amplitude equations are the harmonic balance method, the Galerkin method and the Ritz method. The entire above-mentioned methods make it possible to analyse strongly non-linear systems. Moreover, as it is demonstrated in [3] all of these methods give us identical amplitude equations. The identical amplitude equation is also obtained if the method suggested by Lau *et al.* [12] and the method proposed in [13] are used.

Sometimes, the perturbation methods are used to investigate the free vibrations of large non-linear systems. However, application of these methods is restricted to weakly non-linear systems and for this reason they are out of scope of this paper.

Below, the matrix amplitude equation will be briefly derived with a help of the Galerkin method. An approximate solution to motion equation (3) is assumed in the following form:

$$\mathbf{w}(t) = \mathbf{a}_i \cos z_i \mathbf{w}, \quad (11)$$

where $i=1,2,\dots,n$, \mathbf{a}_i = the unknown amplitude vector, z_i = the appropriately chosen natural number.

The assumed solution is approximate and after introducing (11) into the motion equation (3) the residual vector $\mathbf{r}(t)$ is obtained. The Galerkin conditions have the following form:

$$\frac{2}{T} \int_0^T \mathbf{r}(t) \cos z_l \mathbf{w} dt = \mathbf{0}, \quad (12)$$

where $l=1,2,\dots,n$ and $T = 2\pi/\mathbf{w}$ is the fundamental period of free vibration. From (12) the following matrix amplitude equations can be derived:

$$\left(\mathbf{K}_0 - z_i \mathbf{w}^2 \mathbf{M}\right) \mathbf{d}_{il} \mathbf{a}_i + \frac{1}{2} \mathbf{a}_{ijkl} \mathbf{K}_2(\mathbf{a}_i, \mathbf{a}_j) \mathbf{a}_k = \mathbf{0}. \quad (13)$$

The coefficients \mathbf{d}_{il} and \mathbf{a}_{ijkl} appearing in (13) are defined in reference [3].

After introducing notations:

$$\mathbf{H}_{kl} = \frac{1}{2} \mathbf{a}_{ijkl} \mathbf{K}_2(\mathbf{a}_i, \mathbf{a}_j) \mathbf{a}_k, \quad (14)$$

$$\mathbf{a} = \text{col}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k, \dots, \mathbf{a}_n), \quad (15)$$

$$\tilde{\mathbf{M}} = \text{diag} \left[z_1^2 \mathbf{M}, z_2^2 \mathbf{M}, \dots, z_k^2 \mathbf{M}, \dots, z_n^2 \mathbf{M} \right], \quad (16)$$

$$\tilde{\mathbf{K}} = \text{diag} [\mathbf{K}, \mathbf{K}, \dots, \mathbf{K}, \dots, \mathbf{K}], \quad (17)$$

Equation (13) can be rewritten in the following form:

$$(\tilde{\mathbf{K}} + \tilde{\mathbf{H}}(\mathbf{a}) - \mathbf{w}^2 \tilde{\mathbf{M}}) \mathbf{a} = \mathbf{0}. \quad (18)$$

The $\tilde{\mathbf{H}}(\mathbf{a})$ matrix is build from the blocks \mathbf{H}_{kl} defined by (14).

In many instances, only the fundamental harmonic is taken into account in the solution of motion equation. In this case $z_1 = 1$, $n = 1$, $\tilde{\mathbf{K}} = \mathbf{K}_0$, $\tilde{\mathbf{M}} = \mathbf{M}$, $\mathbf{a} = \mathbf{a}_1$, $\mathbf{a}_{1111} = 3/4$,

$$\tilde{\mathbf{H}}(\mathbf{a}) = \mathbf{H}_{11}(\mathbf{a}) = \frac{3}{8} \mathbf{K}_2(\mathbf{a}, \mathbf{a}) \mathbf{a}, \quad (19)$$

and the amplitude equation takes the form:

$$(\mathbf{K}_0 - \mathbf{w}^2 \mathbf{M}) \mathbf{a} + \frac{3}{8} \mathbf{K}_2(\mathbf{a}, \mathbf{a}) \mathbf{a} = \mathbf{0}. \quad (20)$$

It is easy to observe that Equations (10) and (20) differ by the factor 3/4 in the non-linear term.

If the dimension of $\mathbf{w}(t)$ vector is m , the assumed solution of equation of motion contains n harmonics then the dimension of \mathbf{a} vector is mn and is equal to number of the equations resulting from the Galerkin conditions. However, there are $mn+1$ unknowns in the amplitude equations because apart from the vector of amplitudes the frequency of vibration \mathbf{w} is also unknown. The additional condition must be added to Equation (18). These conditions are discussed in the section that follows.

3. NORMALISATION OF AMPLITUDE VECTOR

The amplitude equation (18) can be treated as the matrix equation with parameter and the frequency \mathbf{w} can be chosen as the main parameter. It is obvious that the trivial solution to amplitude equation exists for all $\mathbf{w} \in R$. The non-trivial solution exist if:

$$\det(\tilde{\mathbf{K}} + \tilde{\mathbf{H}}(\mathbf{a}) - \mathbf{w}^2 \tilde{\mathbf{M}}) = 0. \quad (21)$$

In the linear case, natural frequencies of vibration can be determined from condition (21) but in the non-linear case the same is not possible because of the \mathbf{a} vector appearing in (21). However, it is possible to use the condition (21) as the additional equation to the amplitude equation (18). To date, this possibility has not been investigated.

In literature, one can also find other proposal for the above-mentioned condition, which is can be understood simply as a norm of the amplitude vector. According to the simplest and most popular condition, the freely chosen element of \mathbf{a} vector (say r and usually the

largest one) has a prescribed value, which can be written as:

$$a_r = \mathbf{a}, \quad (22)$$

where \mathbf{a} = the maximum norm of amplitude vector. The condition (22) is used in [6-11]. In the paper [3], the additional equation is taken in the form:

$$\mathbf{a}^T \mathbf{a} = \mathbf{a}^2, \quad (23)$$

where \mathbf{a} = the assumed norm of amplitude vector.

In the paper [13], the \mathbf{a} vector is normalised in such a way that the average kinetic energy of the system of interest has a given value R and the above-mentioned condition can be written in the form:

$$\frac{1}{2} z_i^2 \mathbf{a}_i^T \mathbf{M} \mathbf{a}_i = R. \quad (24)$$

A different condition, based on the principle of energy conservation, has recently been proposed and used in a number of papers by Benamar and his co-workers [14-18]. This condition states that the maximum values of the strain energy $U_{\max}(t_1)$ and the kinetic energy $K_{\max}(t_2)$ are equal i.e.

$$U_{\max}(t_1) = K_{\max}(t_2). \quad (25)$$

After inserting the assumed solution (11) into (5) and (6) the condition (25) can be written in the form:

$$\mathbf{w}^2 = \frac{\mathbf{d}_{ij} \mathbf{a}_i^T \mathbf{K}_0 \mathbf{a}_j + \frac{1}{4} \mathbf{m}_{ijkl} \mathbf{a}_i^T \mathbf{K}_2(\mathbf{a}_j, \mathbf{a}_k) \mathbf{a}_l}{z_i z_j \mathbf{d}_{ij} \mathbf{a}_i^T \mathbf{M} \mathbf{a}_j}, \quad (26)$$

where $\mathbf{m}_{ijkl} = 1$ for all i, j, k, l .

In the papers mentioned above, relation (26) is used to remove the natural frequency of vibration from the amplitude equation (18). However, after such modification, the amplitude equation is still homogenous. For this reason, the condition (22) is included into the set of problem equations and simultaneously the equation corresponding to amplitude r is removed from the matrix amplitude equation. Please note that condition (26) looks like the non-linear Rayleigh quotient.

4. NON-LINEAR RAYLEIGH QUOTIENT

The classical Rayleigh quotient can be derived in a few different ways, which will be briefly discussed below. The amplitude equation for the linear system has a form:

$$(\mathbf{K} - \mathbf{w}^2 \mathbf{M}) \mathbf{a} = \mathbf{0}. \quad (27)$$

By pre-multiplying Equation (27) by \mathbf{a}^T we obtain the well known Rayleigh quotient

$$\mathbf{w}^2 = \frac{\mathbf{a}^T \mathbf{K} \mathbf{a}}{\mathbf{a}^T \mathbf{M} \mathbf{a}}. \quad (28)$$

From the mathematical point of view the Rayleigh quotient can be understood as the solution of the

overdetermined set of equations with respect to \mathbf{w} (here the matrix amplitude equation) if the approximation of the amplitude vector is known.

The identical Rayleigh quotient is obtained from the principle of energy conservation, written in the form of relation (25) because for linear systems

$$K_{\max} = \frac{1}{2} \mathbf{w}^T \mathbf{a}^T \mathbf{M} \mathbf{a}, \quad U_{\max} = \frac{1}{2} \mathbf{a}^T \mathbf{K} \mathbf{a}. \quad (29)$$

The considered systems are autonomous and the principle of energy conservation must be satisfied in every time instance. Therefore, the integral of action

$$J = \frac{2}{T} \int_0^T [U(t) - K(t)] dt, \quad (30)$$

can be understood as a functional which express the principle of energy conservation in a weak sense.

Introducing the solution of motion equation of linear systems $\mathbf{w}(t) = \mathbf{a} \cos \mathbf{w}$ into (30), we obtain

$$J(\mathbf{a}, \mathbf{w}) = \frac{1}{2} \mathbf{w}^2 \mathbf{a}^T \mathbf{M} \mathbf{a} - \frac{1}{2} \mathbf{a}^T \mathbf{K} \mathbf{a}. \quad (31)$$

If $J(\mathbf{a}, \mathbf{w})$ will be considered as a functional of \mathbf{a} and \mathbf{w} as the parameter then from the stationary condition $\mathbf{d}J = 0$ Equation (27) is obtained. Moreover, it is assumed that the value of functional $J(\mathbf{a}, \mathbf{w})$ at the point of stationarity is equal zero, which means that the condition of conservation of average energy is exactly fulfilled. Once again, from the condition $J(\mathbf{a}, \mathbf{w}) = 0$ we obtain the Rayleigh quotient. In conclusion, we have three ways of derivation of the Rayleigh quotient.

Up to now, the Rayleigh quotient has not been defined for non-linear systems. Below, the possibilities of extending of this quotient for non-linear systems of interest will be discussed.

By pre-multiplying the amplitude equation (13) by \mathbf{a}^T the following quotient

$$\mathbf{w}^2 = \frac{\mathbf{d}_i \mathbf{a}_i^T \mathbf{K}_0 \mathbf{a}_i + \frac{1}{2} \mathbf{a}_{ijkl} \mathbf{a}_i^T \mathbf{K}_2(\mathbf{a}_i, \mathbf{a}_j) \mathbf{a}_k}{z_i \mathbf{d}_i \mathbf{a}_i^T \mathbf{M} \mathbf{a}_i}, \quad (32)$$

is obtained. If only one harmonic is included in the solution of motion equation, then

$$\mathbf{w}^2 = \frac{\mathbf{a}^T \mathbf{K}_0 \mathbf{a} + \frac{3}{8} \mathbf{a}^T \mathbf{K}_2(\mathbf{a}, \mathbf{a}) \mathbf{a}}{\mathbf{a}^T \mathbf{M} \mathbf{a}}. \quad (33)$$

Following, in the same way, with the amplitude equation (10), resulting from the collocation condition, we obtain:

$$\mathbf{w}^2 = \frac{\mathbf{a}^T \mathbf{K}_0 \mathbf{a} + \frac{1}{2} \mathbf{a}^T \mathbf{K}_2(\mathbf{a}, \mathbf{a}) \mathbf{a}}{\mathbf{a}^T \mathbf{M} \mathbf{a}}. \quad (34)$$

Quotients (32), (33) and (34) will be called the non-linear Rayleigh quotients of first kind.

Possibilities resulting from the condition of energy conservation will be now analysed in detail. Please note that Equation (26) resulting from (25) can be

understood as the second possible definition of the non-linear Rayleigh quotient. This quotient differs with one given by (32) because, in general, $\mathbf{m}_{jkl} \neq \mathbf{a}_{ijkl}$. Therefore, quotients resulting from energy conditions will be called the non-linear Rayleigh quotients of second kind.

If one harmonic solution of motion equation is of interest then $n=1$, $\mathbf{a}_1 = \mathbf{a}$, $z_1 = 1$ and from (26) we obtain the following quotient:

$$\mathbf{w}^2 = \frac{\mathbf{a}^T \mathbf{K}_0 \mathbf{a} + \frac{1}{4} \mathbf{a}^T \mathbf{K}_2(\mathbf{a}, \mathbf{a}) \mathbf{a}}{\mathbf{a}^T \mathbf{M} \mathbf{a}}, \quad (35)$$

which differs both from quotient (33) and (34).

Furthermore, the non-linear Rayleigh quotient can be defined on the basis of the principle of energy conservation in an average sense. This principle is written in the form:

$$U_a = K_a, \quad (36)$$

where K_a = the average kinetic energy and U_a = the average strain energy. These energies are defined as:

$$K_a = \frac{2}{T} \int_0^T K(t) dt, \quad U_a = \frac{2}{T} \int_0^T U(t) dt. \quad (37)$$

It is obvious that the considered system is autonomous and that the principle of energy conservation can be satisfied in every time instance t . However, we must take into account that the exact solution of motion equation is not known. Neither the principle of energy conservation nor the equation of motion are fulfilled by the approximate solution in every time instance t .

It seems to be reasonable that for the approximate solutions, the principle of energy conservation must be fulfilled in an average sense.

After introducing Equation (11) into relations (37), (5) and (6) we obtain:

$$K_a = \frac{1}{2} z_i z_j \mathbf{d}_{ij} \mathbf{w} \mathbf{a}_i^T \mathbf{M} \mathbf{a}_j, \quad (38)$$

$$U_a = \frac{1}{2} \mathbf{d}_{ij} \mathbf{a}_i^T \mathbf{K}_0 \mathbf{a}_j + \frac{1}{8} \mathbf{a}_{ijkl} \mathbf{a}_i^T \mathbf{K}_2(\mathbf{a}_j, \mathbf{a}_k) \mathbf{a}_l, \quad (39)$$

From condition (36) we obtain the following non-linear Rayleigh quotient:

$$\mathbf{w}^2 = \frac{\mathbf{d}_{ij} \mathbf{a}_i^T \mathbf{K}_0 \mathbf{a}_j + \frac{1}{4} \mathbf{a}_{ijkl} \mathbf{a}_i^T \mathbf{K}_2(\mathbf{a}_j, \mathbf{a}_k) \mathbf{a}_l}{\mathbf{d}_{ij} \mathbf{a}_i^T \mathbf{M} \mathbf{a}_j}. \quad (40)$$

Once again, we obtain another formula for the non-linear Rayleigh quotient.

In the linear case, the Rayleigh quotient can be derived from the condition $J(\mathbf{a}, \mathbf{w}) = 0$. Now, we will analyse consequences of this condition in a non-linear case. Considerations are restricted to the one harmonic solution of motion equation, i.e. when $\mathbf{w}(t) = \mathbf{a} \cos \mathbf{w}$. Two cases are considered. In the first case the

functional $J(\mathbf{a}, \mathbf{w})$ results from the principle of average energy conservation i.e.

$$J(\mathbf{a}, \mathbf{w}) \equiv J_a(\mathbf{a}, \mathbf{w}) = U_a(\mathbf{a}) - K_a(\mathbf{a}, \mathbf{w}), \quad (41)$$

while in the second case the functional is defined as

$$J(\mathbf{a}, \mathbf{w}) \equiv J_m(\mathbf{a}, \mathbf{w}) = U_{\max}(\mathbf{a}) - K_{\max}(\mathbf{a}, \mathbf{w}). \quad (42)$$

Inserting Equation (10) into (5) and (6) and using relations (25) and (37) we can write:

$$J(\mathbf{a}, \mathbf{w}) = \frac{1}{2} \mathbf{a}^T \mathbf{K}_0 \mathbf{a} + \frac{1}{8} \mathbf{k} \mathbf{a}^T \mathbf{K}_2(\mathbf{a}, \mathbf{a}) \mathbf{a} - \frac{1}{2} \mathbf{w}^2 \mathbf{a}^T \mathbf{M} \mathbf{a}, \quad (43)$$

where $\mathbf{k} = 3/4$ in the first case and $\mathbf{k} = 1$ in the second. Now \mathbf{a} is the trial vector and \mathbf{w} is treated as a parameter.

At a stationary point of $J(\mathbf{a}, \mathbf{w})$ the variation $dJ(\mathbf{a}, \mathbf{w})$ must vanish, which yields the following stationary condition

$$\mathbf{K}_0 \mathbf{a} + \frac{1}{2} \mathbf{k} \mathbf{K}_2(\mathbf{a}, \mathbf{a}) \mathbf{a} - \mathbf{w}^2 \mathbf{M} \mathbf{a} = \mathbf{0}, \quad (44)$$

which agrees with the previously derived amplitudes equation (20) (first case) or (10) (second case). If we additionally introduce the condition that at the point of stationarity $J(\mathbf{a}, \mathbf{w}) = 0$ then the parameter \mathbf{w} can be calculated from the following relation

$$\mathbf{w}^2 = \frac{\mathbf{a}^T \mathbf{K}_0 \mathbf{a} + \frac{1}{4} \mathbf{k} \mathbf{a}^T \mathbf{K}_2(\mathbf{a}, \mathbf{a}) \mathbf{a}}{\mathbf{a}^T \mathbf{M} \mathbf{a}}, \quad (45)$$

which for $\mathbf{k} = 3/4$ agrees with (40) and for $\mathbf{k} = 1$ is identical with (35).

5. REMARKS ON AMPLITUDE EQUATION AND NON-LINEAR RAYLEIGH QUOTIENT

The following conclusions result from the above considerations.

The amplitudes equation (10) resulting from the collocation condition is simultaneously the condition of stationarity of the functional $J_m(\mathbf{a}, \mathbf{w})$ while the amplitude equation (13) derived with a help of the Galerkin method is also the stationary condition of functional $J_a(\mathbf{a}, \mathbf{w})$ resulting from the principle of conservation energy in an average sense.

We have two possible ways of extension of the non-linear Rayleigh quotient concept in a non-linear case. The first way, leading to the non-linear Rayleigh quotient of first kind can be understood as the condition of fulfilling the amplitude equation with respect to \mathbf{w} if some approximation of the \mathbf{a} vector is available. In the second way, the frequency of vibration is calculated from the principle of energy conservation (in an averaging sense or in a sense of conservation of maximal energies).

From the mathematical point of view the considered problem is fully defined by the amplitude equation and the condition of existence of non-trivial solutions (21). Condition (21) is hard to use in computing procedures,

that is why different conditions, such as (22), (23) or (25) are used in practice.

Definitions of the non-linear Rayleigh quotient of first and second kind differ from each other in the non-linear term. Both have clear mathematical or physical interpretation. The Rayleigh quotient of second kind is an additional physical condition which makes the considered problem underdetermined because now we have $n+2$ equations (i.e. n amplitude equations, the condition of existence of non-trivial solution and the condition of energy conservation) with only $n+1$ unknowns. The Rayleigh quotient of first kind can be interpreted as an additional condition from which the best approximation of frequency of vibration can be determined for a given approximation of non-linear eigenvector \mathbf{a} . This condition does not introduce any additional equation into problem formulation and it is only an auxiliary equation, which can be used in the procedure for solving the amplitude equation.

The principle of energy conservation is an essential that should be fulfilled for every autonomous dynamic system. Differences in the proposed non-linear Rayleigh quotients suggest that the approximate solution of motion equation (11) do not fulfil the principle of energy conservation.

In this paper, two formulations of amplitude equation are discussed. The first formulation uses the collocation condition while the second formulation utilises some averaging procedure. Both formulations must be internally consistent. This will be achieved if we will use the functional, the amplitude equation and the non-linear Rayleigh quotient in forms shown in Table 1. Others formulations, including the formulation proposed in papers [14-18] are internally inconsistent. Obviously, this conclusion applies also to approximate solutions with many harmonics.

Both internally consistent formulations are approximate, which means that all solutions are burdened with errors. The rational choice of formulation requires an analysis of accuracy of the results obtained. It is very difficult in the general case, therefore only numerical results of accuracy analysis for beam structures will be presented and briefly discussed in this paper.

Table 1

Formulation	Collocation	Galerkin
Functional	$J_m(\mathbf{a}, \mathbf{w})$	$J_a(\mathbf{a}, \mathbf{w})$
Amplitude equation	Formula (10)	Formula (13)
Rayleigh		

quotient of first kind	Formula (34)	Formula (32)
Rayleigh quotient of second kind	Formula (26)	Formula (40)

6. THE VECTORS ITERATION METHOD

In this section the method to solve amplitude equations using the non-linear Rayleigh quotient is described. The method will be referred as the vector iteration method because of similarities between the algorithm of the proposed method and the well-known method to solve linear eigenvalue problems. In this paper only the procedure of determination of the fundamental frequency and the mode of vibration will be described.

As it was shown above, analysis of non-linear free vibration problem requires solution of the matrix amplitude equation, which can be rewritten here in the following form:

$$\mathbf{F}(\mathbf{a}, \mathbf{a}) = \mathbf{w}^2 \mathbf{M} \mathbf{a}, \quad (46)$$

where $\mathbf{F}(\mathbf{a}, \mathbf{a})$ = the vector whose elements are non-linear functions of elements of \mathbf{a} and the parameter \mathbf{a} (\mathbf{a} = the amplitude of vibration at the point p). It is assumed that the solution exist,

$$\mathbf{F}(\mathbf{0}, \mathbf{a}) = \mathbf{0}, \quad \|\mathbf{a}\|_{\infty} = \max_j a_j = 1, \quad \mathbf{G}(\mathbf{a}, \mathbf{a}) = \frac{\partial \mathbf{F}}{\partial \mathbf{a}} \quad (47)$$

and the $\mathbf{G}(\mathbf{a}, \mathbf{a})$ matrix is positive definiteness.

The solution of amplitude equation presented in this section has been obtained by means of the vectors iteration method. It is a well-known method to solve linear eigenproblems [1], and is extended here for the non-linear case. The procedure concerns the case $\mathbf{a} = const.$ but it is possible to find the backbone curve $\mathbf{w}(\mathbf{a})$ by repeating the calculations for a set of \mathbf{a} values.

The algorithm of the method has an iterative character. It is assumed that certain approximation for $\mathbf{w}(\mathbf{a})$ and \mathbf{a} is known and denoted as \mathbf{w}_i and \mathbf{a}_i (i means the number of iteration). The good first approximation is a non-linear eigenpair obtained for the previous \mathbf{a} value or linear ones obtained by solving the linear eigenproblem as follows:

$$(\mathbf{G}(\mathbf{0}, \mathbf{a}) - \mathbf{w}^2 \mathbf{M}) \mathbf{a} = \mathbf{0}, \quad (48)$$

Now, for given \mathbf{w}_i and \mathbf{a}_i , we can solve the following non-linear algebraic system, for instance, by means of the Newton-Raphson method:

$$\mathbf{F}(\mathbf{c}_{i+1}, \mathbf{a}) = \mathbf{w}_i^2 \mathbf{M} \mathbf{a}_i, \quad (49)$$

where \mathbf{c}_{i+1} is now the unknown vector.

As the next approximation of eigenvector \mathbf{a} we take over the vector proportional to \mathbf{c}_{i+1} and whose norm is equal 1. A new approximation of \mathbf{w} is obtained from:

$$\mathbf{w}_{i+1}^2 = \frac{\mathbf{a}_{i+1}^T \mathbf{F}(\mathbf{a}_{i+1}, \mathbf{a})}{\mathbf{a}_{i+1}^T \mathbf{M} \mathbf{a}_{i+1}}, \quad (50)$$

which is the Rayleigh quotient of first kind.

The iterative process is finished if:

$$|\mathbf{w}_{i+1}^2 - \mathbf{w}_i^2| \leq \mathbf{e}_1 \mathbf{w}_{i+1}^2, \quad \|\mathbf{a}_{i+1} - \mathbf{a}_i\|_2 \leq \mathbf{e}_2 \|\mathbf{a}_{i+1}\|_2, \quad (51)$$

where \mathbf{e}_1 and \mathbf{e}_2 = the accuracy of calculation.

7. RESULTS OF EXAMPLE CALCULATIONS

In this paper, conclusions concerning the accuracy of approximate solutions of equations of motion are draw from the values of the energy error introduced below. This definition follows from the integral of action

$$L = \frac{2}{T} \int_0^T [K(t) - U(t) + W(t)] dt, \quad (52)$$

where $W(t)$ = the potential of external loads. If a system is autonomous then from the principle of energy conservation results that $L = 0$. After introducing the approximate solution into the equation of motion the vector of residuals $\mathbf{r}(t)$ is obtained. The $-\mathbf{r}(t)$ vector is treated as the vector of fictitious forces that must load the system if its vibrations are exactly described by the above-mentioned approximate solution of motion equation. The potential of fictitious forces is given by

$$W(t) = -\mathbf{w}^T(t) \mathbf{r}(t), \quad (53)$$

and the average potential W_a is

$$W_a = -\frac{2}{T} \int_0^T \mathbf{w}^T(t) \mathbf{r}(t) dt. \quad (54)$$

The energy error is defined in the form:

$$\Delta E_a = \frac{L_a}{E_a} = \frac{K_a - U_a + W_a}{K_a + U_a}. \quad (55)$$

If the approximate solution of motion equation has the form $\mathbf{w}(t) = \mathbf{a}_l \cos z_l \mathbf{w}$ then

$$W_a = -\mathbf{a}_l^T \mathbf{s}_l, \quad (56)$$

where $l = 1, 2, \dots, n$, and

$$\mathbf{s}_l = \frac{2}{T} \int_0^T \mathbf{r}(t) \cos z_l \mathbf{w} dt . \quad (57)$$

If the Galerkin method is used, then in all cases $W_a = 0$. However, $W_a \neq 0$ if the collocation method and the Benamar method are used.

7.1 Fixed - fixed beam

The backbone curve for fixed-fixed beam is determined using the finite element method. The beam is divided into ten finite elements. The one harmonic solution is assumed. Results of calculation are shown in Figure 1 as the solid and dashed line for the Galerkin and the Benamar method, respectively. The non-dimensional amplitude of vibration means a/i , where a = the amplitude in the middle of the beam and i = the radius of moment of inertia. Differences between results obtained with a help of both methods are significant. In Figure 2, the average energy error of compared methods is also shown. All errors are negative and the modulus of energy error of the Galerkin method is smaller than the energy error of the Benamar method. The Newton-Raphson method is used to solve the matrix amplitude equation. Usually, three iterations are enough to fulfil convergence conditions.

7.2 Multispan beam

The three span, simply supported beam shown is considered. The first and third span has the length two times larger than the second span. The beam is divided into twenty finite elements. The one harmonic solution of motion equation is assumed. The results of calculation are shown in Figure 3. The

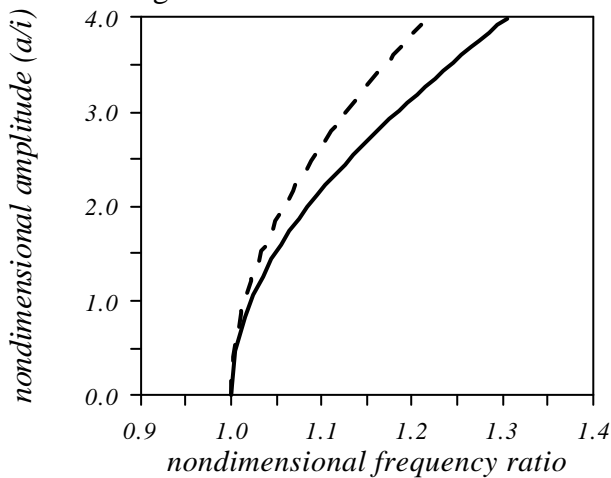


Figure 1. Backbone curves for fixed-fixed beam

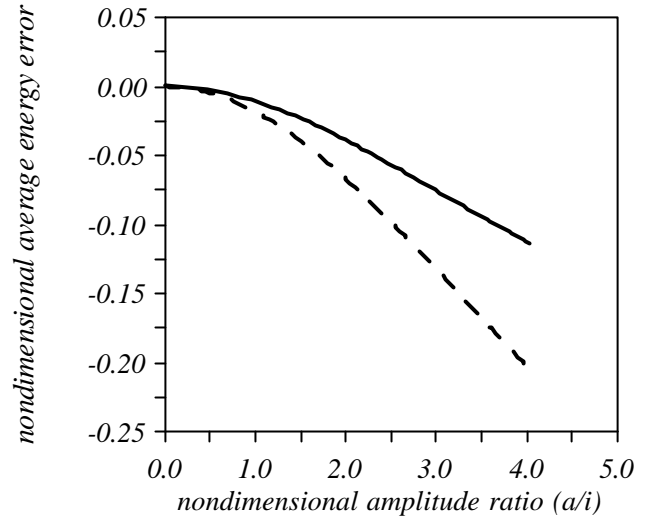


Figure 2. Energy errors of approximate solutions

non-dimensional, fundamental frequency of vibration \mathbf{w}/\mathbf{w}_l versus the non-dimensional amplitude of vibration a/i in the middle of the first span of beam is presented. Results obtained by means of the Galerkin and the collocation method are shown as the solid and dashed line, respectively. Moreover, the vector iteration method is used to determine the mentioned backbone curve. If the Rayleigh quotation of first kind is used than this method gives us results also shown on Figure 3. The small circles presents the results of solution of amplitude equations derived with a help of the Galerkin method while small crosses shown results of solution of amplitude equations resulting from the collocation method. It is obvious that both approaches to the solution of amplitude equation give us identical results. However, the iteration process of the vector iteration method does not converge if the Rayleigh quotient of second kind is

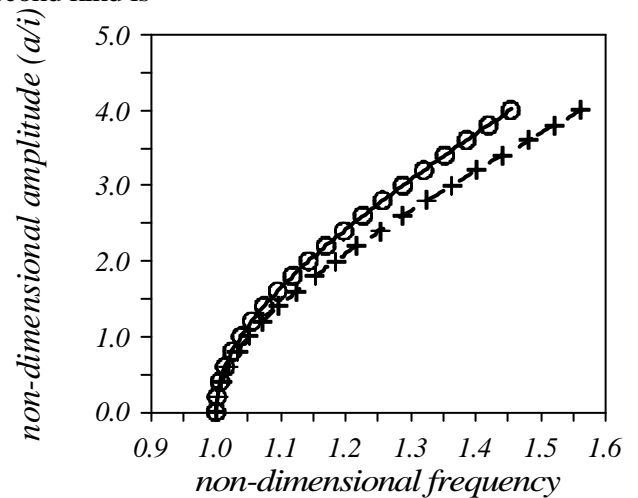


Figure 3. Backbone curves for three span beam

used. These results are in agreement with remarks presented in Section 5.

8. CONCLUDING REMARKS

The following conclusions can be formulated on the basis of considerations presented in this paper.

Several aspects concerning formulation and solution of amplitude equation arising in problems of non-linear free vibration of system with cubic non-linearity are discussed.

The concept of the Rayleigh quotient is extended to the non-linear case. The Rayleigh quotient of first kind is proposed as a right definition in a non-linear case. Moreover, the quotient is used to develop a new method for solving the non-linear eigenvalue problem. The mentioned method is an extension of the vector iteration method to the non-linear case.

Calculation results suggest that the Galerkin method give us most accurate results in comparison with ones obtained using the collocation method and the method proposed in papers [14-18].

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