NON-LINEAR FREE VIBRATIONS OF BEAMS
BY THE FINITE ELEMENT AND
CONTINUATION METHODS

R. Lewandowski

Technical University of Poznań, ul. Piotrowo 5, 60-965 Poznań, Poland

(Received 2 August 1991, and in final form 19 June 1992)

A computational method is presented for determining backbone curves of arbitrary geometry. As an example, beam vibration is considered. The beam response is expanded into a truncated Fourier series with respect to time. After starting from the integral of action, the variational approach and the finite element method are used to formulate the non-linear eigenvalue problem. The continuation method is adopted to solve the resulting non-linear eigenvalue problem and to obtain the non-linear frequencies and modes of vibration. Numerical results for various beams are presented and compared with available results to demonstrate the accuracy and applicability of the method. Moreover, the bifurcation points on some beams’ backbone curves are found and reported for the first time.

1. INTRODUCTION

Natural vibrations of undamped, non-linear systems are of primary importance in studying resonance phenomena because the backbone curves (the amplitude–frequency relations) and the modes of vibrations, i.e., the dynamic characteristics of the systems, are determined. Analytical expressions for the backbone curves are available only for simple systems and, therefore, numerical methods are necessary when considering more complex cases.

The equation of motion for beams with immovable supports is non-linear due to the significant influence of axial forces. The non-linear free vibration of beams has been studied by many researchers in the past. Both a continuum approach [1, 2] and the finite element method [3–5] have been used. Most often, solutions with only one harmonic are taken into account. In this case the backbone curves have rather a simple shape, similar to a parabola, and can be constructed by numerical methods such as the vector iteration method [6], and others suggested by Mei [3, 5] and Welford et al. [7].

In some cases, such as those of internal resonances, the solution with one harmonic is not suitable and may lead to significant errors. Internal resonance is the particular phenomena which arises in non-linear multi-degree-of-freedom systems when the natural frequencies become commensurable [8]. The commensurability of natural frequencies causes, generally speaking, coupling of the normal modes and results in multi-mode and multi-frequency response. This is characteristic of internal resonance. The commensurability condition is satisfied only in a narrow range and consequently the backbone curves now have a more complex shape owing to the presence of sharp peaks, loops and rapidly changing slopes. It is difficult to determine these types of backbone curves by the methods mentioned above.
The perturbation method can be used to study weakly non-linear problems. This method is simple to use if the first order approximate solutions are considered. Other approximate methods such as the Galerkin method, the Ritz method, the averaging method and the harmonic balance method make possible the determination of multi-harmonic solutions for highly non-linear systems. The most popular one is the harmonic balance method. Lau et al. [9] considered free vibration of plates by the incremental harmonic balance method. The method successfully predicts the backbone curve in an incremental manner by incrementing alternatively the characteristic amplitudes or frequency. A similar method has been proposed by Leung [10] who used the increments of the harmonic coefficients of the displacement amplitudes to find new equilibrium states along the backbone curve.

This paper presents a computational algorithm to obtain a backbone curve of arbitrary geometry. Beam vibrations are considered as examples. The time dependent displacements are expressed in terms of a Fourier series with respect to time and this is then truncated to a finite number of harmonics. After starting from the integral of action, the variational approach and the finite element method are used to formulate the non-linear eigenvalue problem. The continuation method is adopted to solve the non-linear eigenvalue problem and consequently to determine the backbone curve and the non-linear modes of vibration. The Newton method and the concept of an "arc-length" constraint equation are used to limit the initial frequency increment and to enable one to trace the complex solution paths. Automatic generation of the backbone curve is possible.

Symmetry of the tangential matrix is preserved during the whole calculation process. Because only the usual matrices like those of mass, stiffness and geometry are used in the formulation, the method can be relatively easily implemented in a standard finite element code.

Finally, numerical examples are given and compared with available results to show the accuracy and usefulness of the proposed method. It is found that the method predicts the backbone curve for internal resonances in a fully automatic way and without any difficulties.

2. FORMULATION

The non-linear strain–displacement relations of the beam are

$$\varepsilon = u_x + \frac{1}{2}(w_x)^2, \quad \kappa = w_{xx},$$

where $(.)_x = d/dx$, and $\varepsilon$, $\kappa$, $u$ and $w$ denote the axial strain, the curvature, and the axial and transverse displacements, respectively. The axial force is given by

$$N(x, t) = EA(x)\varepsilon,$$

where $A(x)$ is the cross-sectional area and $E$ is the elastic modulus.

Neglecting the longitudinal inertia forces and assuming immovable ends of a beam one has [5]

$$\int_0^l u_x \, dx = 0, \quad N(x, t) = N(t) = \frac{EA}{2ld} \int_0^l w_x^2 \, dx,$$

where $A$ and $l$ denote the reference area of cross-section and the length of beam, respectively. Moreover,

$$d = \frac{A}{l} \int_0^l \frac{dx}{A(x)}.$$

Averaging of the axial force over the beam for large amplitude vibration is possible if the beam is slender [11].
Upon taking into account the above results and neglecting the rotary inertia forces, the strain energy $U$ and the kinetic energy $K$ can be written in the forms

$$
U = \frac{1}{2} \int_0^l EI(x)w_x'^2 \, dx + \frac{1}{2} \int_0^l \frac{N^2(t)}{EA(x)} \, dx, \quad K = \frac{1}{2} \int_0^l m(x)w^2 \, dx,
$$

where $I(x)$ is the moment of inertia and the dot denotes differentiation with respect to time.

It is assumed that for the typical finite element shown in Figure 1 the approximate solution has the form

$$
w(x, t) = N(x)w_e(t),
$$

where $N(x)$ denotes the vector of shape functions and $w_e(t)$ is the vector of nodal parameters:

$$
N(x) = \begin{pmatrix} N_1(x), N_2(x), N_3(x), N_4(x) \end{pmatrix},
$$

$$
w_e(t) = \begin{pmatrix} w_a, \varphi_a, w_b, \varphi_b \end{pmatrix},
$$

$$
N_1(x) = 1 - 3\eta^2 + 3\eta^3, \quad N_2(x) = \eta(\eta - 2\eta^2 + \eta^3),
$$

$$
N_3(x) = 3\eta^2 - 2\eta^3, \quad N_4(x) = \eta^3(\eta - \eta^2), \quad \eta = x/l_a.
$$

The average axial force written in matrix notation takes the form

$$
N(t) = \frac{EA}{2ld} \sum_{i=1}^n w_i B_e w_e = \frac{EA}{2ld} w^Bw_e,
$$

where $n$, $B$ and

$$
B_e = \int_0^l N_\alpha N_\alpha' \, dx,
$$

are the total number of elements, the global geometric matrix and the element geometric matrix, respectively. Moreover, $w$ denotes the global vector of nodal parameters.

The integral of action written in matrix form is

$$
W = \frac{2}{T} \int_0^T (K - U) \, dt = \frac{1}{T} \int_0^T \left( w^Mw - w^Mw - \frac{ld}{EA} N^2(t) \right) \, dt,
$$

Figure 1. Typical finite element.
where $T = 2\pi/\omega$ is the unknown period of vibration and $\omega$ is the non-linear frequency, and $M$ and $K$ are the global mass and stiffness matrices. As usual, their elemental counterparts are defined by

\[ M_i = \int_{0}^{t} m(x)N_i(x)N_i(x) \, dx, \quad K_{ij} = \int_{0}^{t} EI(x)N_{i,x}N_{j,x} \, dx. \]  

(9)

Hamilton's principle leads to the equation of motion [5]:

\[ M\ddot{w} + Kw + N(t)Bw = 0. \]  

(10)

Notice that the non-linearity in the beam equation of motion is cubic with respect to displacement.

The periodic solution is expanded into the Fourier series

\[ w(t) = \sum_{l=1}^{n} q_l(t), \]  

(11)

where $l = 1, 2, \ldots, n$ and the summation convention is adopted for repeated indices.

The functions $q_l(t)$ are assumed to be of the form

\[ q_l(t) = \cos((2l-1)\omega t), \]  

(12)

which means that only the odd harmonics are taken into account. This choice of $q_l(t)$ is supported by results obtained with the help of the perturbation method [12]. If the perturbation procedure is applied to determine the solution of a differential equation with cubic non-linearity then only the odd harmonics take part in the solution.

Inserting the suggested form of solution (11) into the integral of action (8) one has, after integration with respect to time, the functional

\[ W = \frac{1}{2}\omega^2\gamma^2 \sum_{l=1}^{n} \int_{0}^{t} M_{ij} \ddot{q}_l(t) \ddot{q}_j(t) \, dt, \]  

(13)

where

\[ C_{ij}(\gamma_1, \gamma_2) = \frac{EA}{2ld} \gamma_1 \gamma_2, \quad \alpha = \frac{2}{T \omega^2} \int_{0}^{t} q_l(t)q_j(t) \, dt, \]  

\[ \beta = \frac{2}{T} \int_{0}^{t} q_l(t)q_j(t) \, dt, \quad \delta_{ij} = \frac{2}{T} \int_{0}^{t} q_l(t)q_j(t)q_k(t)q_i(t) \, dt. \]  

(14)

The functions $q_l(t)$ are orthogonal, which means that

\[ \alpha_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ (2l-1)^2 & \text{if } i = j, \end{cases} \quad \beta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases} \]  

(15)

Taking into account the simple relations

\[ \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) = \frac{1}{2}(\cos \alpha + \cos \beta), \]  

and

\[ J(i) = \frac{2}{T} \int_{0}^{t} \cos 2\omega t \, dt = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases} \]  

(16)

one can express relation (14a) in the form

\[ 4\delta_{ij} = J(i+j+k+l+1) + J(i+j-k-l-1) + J(i+j+k+l-1) \]

\[ + J(i+j-k-l+1) + J(i+j-k+l-1) + J(i+j-k+l+1) \]

\[ + J(i+j-k+l) + J(i+j-k-l). \]  

(17)
The stationary condition $\delta W = 0$ leads to the equation

$$F_1(\omega^2, \nu) = \alpha_1 \omega^2 \mathbf{M} \nu - \beta_2 \mathbf{K} \nu - \delta_{2h} C_{2h} \nu \mathbf{B} \nu = 0,$$

which defines the non-linear eigenvalue problem. The second power of the non-linear frequency is a non-linear eigenvalue and the vector $\nu = \text{col} (\nu_1, \nu_2, \ldots, \nu_n)$ is a generalized non-linear eigenmode.

The matrix notation and summation convention used simultaneously, as presented above, lead to an algebraic equation which can be easily derived and has a very compact form. This is believed to be the advantage with earlier approaches to the problem. After introducing the notation

$$\mathbf{F}(\omega^2, \nu) = \text{col} (F_1(\omega^2, \nu), \ldots, F_n(\omega^2, \nu)),$$

$$\mathbf{M} = \begin{bmatrix} \alpha_{11} \mathbf{M} & \cdots & \alpha_{1n} \mathbf{M} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} \mathbf{M} & \cdots & \alpha_{nn} \mathbf{M} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \beta_{11} \mathbf{K} & \cdots & \beta_{1n} \mathbf{K} \\ \vdots & \ddots & \vdots \\ \beta_{n1} \mathbf{K} & \cdots & \beta_{nn} \mathbf{K} \end{bmatrix},$$

$$\mathbf{B}(\nu) = \begin{bmatrix} \gamma_{11} \mathbf{B} \\ \vdots \\ \gamma_{n1} \mathbf{B} \end{bmatrix}, \quad \gamma_0 = \delta_{2h} C_{2h} \nu,$$

equation (18) can be written in the more compact form

$$\mathbf{F}(\omega^2, \nu) = (\omega^2 \mathbf{M} - \mathbf{K} - \mathbf{B}(\nu)) \nu = 0.$$  

(19)

3. SOLUTION OF THE NON-LINEAR EIGENVALUE PROBLEM

Systems of non-linear equations depending on a parameter naturally arise in many problems: e.g., in plasma physics problems [13], in the non-linear stability theory of elastic structures or in non-linear free vibration problems. There exist some numerical methods which make possible a solution of non-linear eigenvalue problem arising in non-linear dynamics. Mei [3] developed a linearization technique to change the non-linear problem into a sequence of linear problems which afterwards are solved in the standard way. Moreover, this method was extended to a case in which the matrices $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{B}$ in equation (18) are non-linear functions of $\omega$: i.e., when the dynamic finite element method is used in the formulation [5].

Vector iteration methods can also be used to solve non-linear eigenvalue problems arising in non-linear dynamics. These methods are, in fact, a simple generalization of the well-known method of the theory of linear eigenproblems [14] to non-linear cases. Detailed descriptions have been given in references [6, 15].

A different method was proposed by Wellford et al. [7]. A constraint equation is added to the system of non-linear equations describing the non-linear eigenvalue problem. The extended system of equations is solved by the Newton method. In this method the constraint equation is expressed by the total eigenvector and, consequently, the backbone curve of complex shape cannot be determined.

In references [9, 10] the incremental form of constraint equation has been used. Lau et al. [9] used the increments of the characteristic amplitudes or frequency in the constraint equation. Leung [10] suggested to use alternatively the increments of the harmonic coefficients of the displacement amplitudes or frequency for controlling the total displacement amplitudes and frequency increments. These methods successfully determine backbone curves of complex shape. The shortcoming of the method is the necessity for modification of the tangential matrix during an iteration process.
In this paper the “arc-length” constraint equation, originally proposed by Crisfield [16] and the Newtonian algorithm, are adopted to solve the non-linear eigenvalue problem. The resulting incremental–iterative algorithm is general and systematic, preserves the symmetry of the tangential matrix in the whole iteration process and can be relatively easy adopted in a standard finite element code.

The non-linear eigenvalue problem (19) has the trivial solution \( \nu = 0 \) for all \( \omega \in R \). The branching from the trivial solution is called primary bifurcation, while the branching from a non-trivial solution is called secondary bifurcation or simply bifurcation. An element \((\nu, \omega^2)\) of a branch of solutions of \( F(\omega^2, \nu) = 0 \) is called an ordinary point if the matrix of the first derivative of \( F \) with respect to \( \nu \), denoted here by \( D \), is not singular. In the opposite case a pair \((\nu, \omega^2)\) is called an exceptional point (turning or bifurcation point). The mathematical foundation and methods of the bifurcation theory have been described in reference [17].

It is assumed that the unknown equation of the backbone curve is a continuous function of the generalized non-linear eigenmode \( \nu \) and the natural frequency \( \omega \). Moreover, the solution described by \( \nu^*, \omega^* \) and connected with the point \( m \) on the backbone curve is known. The incremental process is started from a chosen primary bifurcation point. Then the next one on this curve, which is at a fixed distance \( s \) from the point \( m \), can be determined. Denoting by \( \Delta \nu \) the total increment of \( \nu \) a constraint equation is written in the form

\[
(\nu - \nu^*)(\nu - \nu^*) = \Delta \nu \Delta \nu = s^2. \tag{20}
\]

The relation (20) together with equation (19) are a coupled system of equations in which \( \nu \) and \( \omega \) are unknown quantities.

The incremental form of equation (19) can be expressed as

\[
D \delta \nu = \tilde{M} \delta \nu \delta \omega^2 - \tilde{F}, \tag{21}
\]

where \( \delta \nu \) is the iterative change of \( \nu \) and

\[
D = (\omega^2 \tilde{M} - \tilde{K} - \tilde{B}(\nu) - \tilde{E}(\nu)),
\]

\[
\tilde{E}(\nu) = \begin{bmatrix}
E_{11} & \ldots & E_{1p} & \ldots & E_{1m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
E_{p1} & \ldots & E_{pp} & \ldots & E_{pm} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
E_{m1} & \ldots & E_{mp} & \ldots & E_{mm}
\end{bmatrix},
\]

\[
E_\nu(\nu) = (EA/2l)dB(\delta \nu_0 \nu_1\nu_2^2 + \delta \nu_0 \nu_1\nu_2^1)B. \tag{22}
\]

The solution of equation (21) can be written in the form

\[
\nu_{i+1} = \nu_0 + \Delta \nu_i + \delta \nu_i, \quad \omega^2_{i+1} = \omega^2_0 + \Delta \omega^2_i + \delta \omega^2_i, \quad \delta \nu = \delta \omega^2 \delta \nu_1 + \delta \nu_2, \quad \delta \nu_1 = D^{-1} \omega^2 \tilde{M} \nu, \quad \delta \nu_2 = -D^{-1} \tilde{F}, \quad \Delta \nu_{i+1} = \Delta \nu_i + \delta \nu_i, \tag{23}
\]

where \( i \) denotes the number of the iteration. Substituting \( \Delta \nu_{i+1} \), from equations (23) into the constraint equation (20) gives the relation for \( \delta \omega^2 \)

\[
a_1 (\delta \omega^2)^2 + a_2 \delta \omega^2 + a_3 = 0, \tag{24}
\]

where

\[
a_1 = \delta \nu_1 \delta \nu_2, \quad a_2 = 2(\Delta \nu_1 + \delta \nu_2) \delta \nu_1, \quad a_3 = (\Delta \nu_1 + \delta \nu_2)^2 (\Delta \nu_1 + \delta \nu_2) - s^2. \tag{25}
\]
Since $\delta y_1$, $A v_i$ and $\delta y_2$ are all known, equation (24) can be easily solved. Selection of an appropriate root was proposed by Crisfield [16] to avoid doubling back on the response curve. The "angle" $\beta$ between the incremental amplitude vector $A v_i$, before the present iteration, and the one $A v_{i+1}$ after the current iteration, should be positive:

$$\beta = A v_{i+1} \cdot A v_i.$$  \hspace{1cm} (26)

The two roots $\delta \omega^2$ give two values for $\beta$ and the appropriate root is that which gives a positive "angle". If both the "angles" are positive the appropriate root is that which is the closest to the linear solution of equation (21), which is

$$\delta \omega^2 = -a_1/a_2.$$  \hspace{1cm} (27)

In some cases equation (24) gives complex roots and a problem arises of choosing the proper root. If the roots, in the Crisfield method, become complex the incremental length $s$ is reduced by half and the iteration process is restarted from the previous known point $m$.

The iterations are repeated until the inequalities

$$|\delta \omega^2_{i+1} - \delta \omega^2_i| \leq \epsilon_1 |\delta \omega^2_{i+1}|, \quad A v_{i+1} \cdot A v_i \leq \epsilon_2 A v_{i+1}, \quad F_{i+1} \leq \epsilon_3 \omega^2_{i+1} \delta y_1^2,$$  \hspace{1cm} (28)

are satisfied ($\epsilon_1 = \epsilon_2 = \epsilon_3 = 0.001$ in the present work).

Now, the sign of the matrix $D$ is also calculated to check the existence of bifurcation points on the solution path. If the signs of the matrix $D$ for two successive $\omega$, i.e., $\omega$ and $\omega + 1$, are different, this means that there exists a particular $\tilde{\omega}$ for which the matrix $D$ is singular and that a bifurcation or turning point occurs. Up to now only the fundamental solution paths have been mainly determined in the numerical implementation. Determination of the secondary solution paths is not considered here in a systematic way. However, some simple procedures, successfully used recently, will be briefly described in the next section. This problem needs further theoretical and numerical studies.

So far, it has been assumed that the incremental length $s$ is known. The first incremental length is calculated from relation (29)

$$s^2 = v_0^2 v_6,$$  \hspace{1cm} (29)

where $v_6$ is an appropriately scaled linear eigenmode associated with a chosen primary bifurcation point: i.e., a chosen linear frequency of oscillation. The magnitude of the next incremental length is governed by

$$s_j = (I_j/I_{j-1})^{1/2} s_{j-1},$$  \hspace{1cm} (30)

where $s_{j-1}$ is the incremental length at the $(j - 1)$th increment, $I_{j-1}$ is the number of the iteration needed at the $(j - 1)$th increment and $I_j$ is the desired number of iterations (five or six in the present work).

If the convergence criteria are not achieved within an assumed maximum number of iterations, the incremental length is reduced by half. In the same way the incremental length is reduced if the modulus of the total increment $\delta \omega^2$ is greater than an assumed value $\delta \omega^2$. The initial value of $\delta \omega^2$ is obtained from

$$\delta \omega^2_0 = \pm \sqrt{(\delta v_1 \delta v_i)^2},$$  \hspace{1cm} (31)

and the sign in equation (31) is chosen as the sign of the total frequency increment $\delta \omega^2$ in the previous incremental step.
Moreover, a numerical relaxation is applied if the increment of frequency is oscillating. If the conditions of oscillations

\[ \delta \omega_{1+1} \delta \omega_1 < 0, \quad |\delta \omega_{1+1}| < |\delta \omega_1| \]  

(32)

are satisfied, then \( \delta \omega \) and \( \delta v \) are scaled by the relaxation factor \( \alpha = 0.5 \) in the following way:

\[ \delta \bar{\omega}^2 = \alpha \delta \omega^2, \quad \delta \bar{v} = \alpha (\delta \omega^2 \delta v_1 + \delta v_2). \]  

(33)

This simple and numerically efficient scheme improves the convergence rate.

4. NUMERICAL EXAMPLES

In the numerical calculations a two harmonic solution has mainly been used. The convergence problem for the truncated Fourier series will not be discussed in general, but some comments concerning the examples examined will be given later. For a two harmonic solution the basic relations, written in section 2 in a general form, are rewritten here for convenience:

\[ w(t) = v_1 \cos \omega t + v_2 \cos 3\omega t, \]  

(11a)

\[ F_1(\omega^2, \nu) = (K - \omega^2M + \frac{1}{2}C_{11}(\nu) + \frac{1}{2}C_{12}(\nu) + \frac{1}{2}C_{22}(\nu))v_1 + \frac{1}{2}C_{11}(\nu) + C_{12}(\nu))Bv_2 = 0, \]  

\[ F_2(\omega^2, \nu) = \frac{1}{2}C_{11}(\nu) + C_{12}(\nu)Bv_1 + (K - 9\omega^2M + \frac{1}{2}C_{22}(\nu) + \frac{1}{2}C_{11}(\nu))Bv_2 = 0. \]  

(18a)

The incremental equations associated with equation (18a) are

\[ D_{11}(\nu)\delta v_1 + D_{12}(\nu)\delta v_2 = M\delta \omega^2 - F_1(\omega^2, \nu), \]  

\[ D_{12}(\nu)\delta v_1 + D_{22}(\nu)\delta v_2 = 9M\nu_1 \delta \omega^2 - F_2(\omega^2, \nu), \]  

(21a)

where

\[ D_{11}(\nu) = K - \omega^2M + \frac{1}{2}C_{11}(\nu) + \frac{1}{2}C_{12}(\nu) + \frac{1}{2}C_{22}(\nu)B \]  

\[ + (EA/2ld)B\left(\frac{1}{3}v_1 v_1 + \frac{1}{3}v_1 v_1 + \frac{1}{3}v_2 v_2 + v_2 v_1 \right)B, \]  

\[ D_{12}(\nu) = D_{21}(\nu) = \frac{1}{2}C_{11}(\nu) + C_{12}(\nu)B + (EA/2ld)B\left(\frac{1}{3}v_1 v_1 + v_1 v_2 + v_2 v_1 \right)B, \]  

\[ D_{22}(\nu) = K - 9\omega^2M + \frac{1}{2}C_{22}(\nu) + \frac{1}{2}C_{11}(\nu)B + (EA/2ld)B\left(\frac{1}{3}v_2 v_2 + v_1 v_1 \right)B. \]  

(22a)

Similar relations for a one harmonic solution are obtained from relations (11a), (18a), (21a), (22a) by setting formally \( v_2 = 0 \).

4.1. EXAMPLE 1

First, the results obtained by different methods and with different number of harmonics used in the approximate solution are compared in Table 1. A hinged–hinged beam of high slenderness ratio with immovable ends is taken as an example. An analytical solution exists in this case. With the solution expressed as

\[ w(x, t) = \omega(t) \sin \left(\pi x / l\right), \]  

(34)

the problem is reduced to the Duffing equation

\[ \omega^2 u_{xx} + \omega^2 (u + \omega^2 u) = 0, \]  

(35)
where $\omega_n$ is the linear frequency of vibration, $\tau = \omega t$, $u = u/r$ and $r^2 = I/A$ is the second power of radius of inertia. The relation between the non-dimensional amplitude $\alpha = \max u(t)$ and the non-dimensional frequency $\omega/\omega_n$ is given by [1]

$$\frac{\omega}{\omega_n} = \frac{\alpha}{\alpha_n} = \frac{\pi \sqrt{1 + \alpha^2/4}}{2} \frac{1}{F(b, \pi/2)}$$  \hspace{1cm} (36)

where $b^2 = 0.5 \alpha^2/(4 + \alpha^2)$ and $F(\cdot)$ represents the elliptic function.

In Table 1 a comparison of the present results with the analytical solution expressed by equation (36) and that previously published in reference [18] is given. The results for the fundamental non-linear frequency are presented and good agreement are observed. Moreover, it is evident that in this case the two harmonic solution leads to sufficiently accurate results. The beam is divided into six equal elements and one or two harmonics are used in the approximate solution. The results for two harmonics and for different numbers of elements are presented in Figure 2. Because of the rapid convergence, a four-element solution is sufficiently accurate for the fundamental mode.

In addition, the non-dimensional values of the fundamental linear frequency $\lambda = \omega_n^2 ml^4/\text{EI}$ have been calculated for a beam divided into different numbers of elements. The results are given in Table 2 where the present ones are compared with previously

<table>
<thead>
<tr>
<th>$\alpha/r$</th>
<th>Reference [18], one harmonic</th>
<th>Reference [18], two harmonics</th>
<th>Reference [1], exact</th>
<th>Present analysis (six elements)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha/\alpha_n$</td>
<td>$\alpha/\alpha_n$</td>
<td>$\alpha/\alpha_n$</td>
<td>$\alpha/\alpha_n$</td>
</tr>
<tr>
<td>1.0</td>
<td>1.029</td>
<td>1.0892</td>
<td>1.0892</td>
<td>0.9815</td>
</tr>
<tr>
<td>2.0</td>
<td>1.3630</td>
<td>1.3178</td>
<td>1.3177</td>
<td>1.0087</td>
</tr>
<tr>
<td>3.0</td>
<td>1.7030</td>
<td>1.6255</td>
<td>1.6256</td>
<td>1.0087</td>
</tr>
</tbody>
</table>

† $\alpha$ is the total amplitude in the middle of the beam.

![Figure 2](image-url)
obtained values of $\lambda$ [3, 4]. The agreement of the present results for six elements with the others is good.

The bifurcation point on the backbone curve is found if the two-harmonic solution is taken into account. In this case the determinant of the matrix $D$ becomes zero for a non-dimensional frequency in the range $1.446 < \omega_1/\omega_1 < 1.451$ (point $a$ in Figure 2). In Figure 2 the amplitude ratio is equal to $|v_1 + v_2|/\sigma$, where $v_1$ and $v_2$ are the amplitudes of the first and second harmonics in the middle of the beam. The frequency ratio is defined as the quotient of the non-linear fundamental frequency and the linear fundamental frequency in all the cases considered.

A simple procedure, frequently used for determining the post-buckling paths for non-linear structures subjected to static loads [19], is adopted here to obtain the above-mentioned secondary part of the backbone curve. The procedure will be briefly described below. A detailed description has been given in reference [19].

Initially the bifurcation point must be determined with appropriate accuracy. Near the bifurcation point the associated eigenvalue problem

$$(D - \mu I)x = 0,$$  \hfill (37)$$

has to be solved in order to calculate the zero eigenvalues $\mu_i$. A case where only one eigenvalue is equal to zero is taken into account in this paper. The eigenvector $x_i$ indicates the direction of the solution to be followed for the calculation of a secondary branch.

<table>
<thead>
<tr>
<th>End conditions</th>
<th>Exact (eight elements)</th>
<th>FEM [3] (eight elements)</th>
<th>FEM [4] (eight elements)</th>
<th>Present analysis (Six elements)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hinged–hinged</td>
<td>97-409</td>
<td>97-412</td>
<td>97-409</td>
<td>97-460</td>
</tr>
<tr>
<td>Fixed–fixed</td>
<td>500-658</td>
<td>500-658</td>
<td>500-564</td>
<td>501-894</td>
</tr>
</tbody>
</table>

Figure 3. Example 1: a projection of the backbone curve for a simply supported beam on the plane $v_1/\sigma, \omega_1/\omega_1$. 
associated with $\mu_i = 0$. Therefore this eigenvector is used for a perturbation of the solution vector $v$ at the bifurcation point. The perturbed solution $\tilde{v}$ is obtained by adding the scaled eigenvector to the vector $v$,

$$\tilde{v} = v + \xi \frac{x_i}{\|x_i\|},$$

(38)

after which it is used as a starting vector for branch switching. In expression (38) $\xi$ denotes a scaling factor; its value is critical for successful branch switching. The magnitude of $\xi$ can be estimated by

$$\xi = \pm \|v\|/\eta,$$

(39)

where $\eta$ is a factor the value of which will be of the order of 100 approximately.

The secondary path of the backbone curve branching from point $a$ has also been determined and is shown, together with the fundamental one, in Figures 3–5. In Figure 3 the projection of the backbone curve on the plane $v_i/r$ and $\omega/\omega_1$ is shown, while
in Figure 4 the same backbone curve is projected on the plane $v_2/r$, $\omega/\omega_1$. In both figures $v_1/r$ and $v_2/r$ are the amplitude ratios at $x=0.25l$, respectively. Moreover, $\omega/\omega_1$ is the frequency ratio. In Figure 5 the projection of the backbone curve on the plane $v_1/r$, $v_2/r$ is also shown, to make the results complete.

The existence of a bifurcation point on the backbone curve for a simply supported beam is closely related with the range where the ratio of the non-linear fundamental frequency and the second linear frequency is approximately equal to 3 (in our case 2.76). Consequently, the bifurcation point indicates that internal resonance may occur if the beam is subjected to a harmonic excitation. It is well known that for a simply supported beam the first and second linear modes are symmetrical and antisymmetrical, respectively. Moreover, it was found that both non-linear modes have exactly the same shape as in the linear case if the one harmonic solution is taken into account [2, 20]. For the

Figure 7. Example 2: the first amplitude ratio $v_1/r$ vs. the frequency ratio $\omega/\omega_1$ in the middle of the hinged-clamped beam.
two-harmonics solution the part denoted by \( \psi \) of the non-linear modes associated with the fundamental path is also symmetrical and is almost equal to the linear one. Furthermore, the vector \( \psi_2 \) is also symmetrical and has negligible values in comparison with \( \psi_1 \). The non-linear modes associated with the secondary path are more complicated. The vector \( \psi_p \), also gives a symmetrical mode, but \( \psi_p \) now gives an antisymmetrical one, and the values of both are comparable (see Figures 3–5). The resultant shape of the deflected beam is asymmetrical and changes during a period of vibration. Because, for a fundamental path, both vectors \( \psi_1 \) and \( \psi_2 \) produced a strictly symmetrical mode then, in a range of internal resonance, a continuous growth of the antisymmetrical part of a mode of vibration is impossible and a bifurcation point occurs. In the opposite case, as will be shown in the next example, a characteristic loop on the backbone curve is observed.

Figure 8. Example 2: the second amplitude ratio \( v_2/r \) vs. the frequency ratio \( \omega_1/\omega_1 \) in the middle of the hinged-clamped beam.

Figure 9. Example 2: a projection of the backbone curve for the hinged-clamped beam on the plane \( v_2/r, v_1/r \).
4.2. EXAMPLE 2

The hinged-clamped beam is considered next because in this case the second linear frequency is almost three times the fundamental one ($\omega_2 = 3.24 \omega_1$) and consequently internal resonance occurs if the first frequency $\omega_1$ increases (due to the growth of amplitude) so that $\omega_1$ becomes equal to $\omega_2/3$. The solution with two harmonics must be taken into account to obtain the solution with sufficient accuracy (see also references [9, 18, 21]). The beam is divided into eight equal elements. The calculated backbone curve is presented in Figure 6, where the non-dimensional amplitude in the middle of the beam is equal to $[v_1 + v_2]/r$. Moreover, the variation of the non-dimensional harmonic amplitudes $v_1/r$ and $v_2/r$ with an increase of the frequency ratio $\omega/\omega_1$ ($\omega_1$ is the fundamental linear frequency) are shown in Figures 7 and 8, respectively. Now, both modes given by vectors $v_1$ and $v_2$ are asymmetrical and in a range of internal resonance both vectors can change continuously. A characteristic loop on the backbone curve is observed. Moreover, the projection of the backbone curve on the plane $v_1/r, v_2/r$ is shown in Figure 9.

4.3. EXAMPLE 3

The three-span beam shown in Figure 10 is taken as an example because in this case internal resonance also occurs. For the length span ratio $l_1/l_2 = 1.773$ the relation between the first and third linear frequencies is $\omega_3 = 3.12 \omega_1$. Moreover, the frequency ratio $\omega_3/\omega_1$ is equal to 3.76. The first and third linear modes are strictly symmetrical and the second and third ones are strictly antisymmetrical. The first and third span is divided into eight elements and the second one into four elements. As in the previous case, the two-harmonics solution is taken into account. The results of calculations are shown in Figures 11–13. In Figures 11 and 12 the first and second amplitude ratios $v_1/r$ and $v_2/r$ are shown vs. the frequency ratio $\omega/\omega_1$ at $x = 0.25l$ in the first span, respectively.

Figure 10. Example 3: the three-span beam.

Figure 11. Example 3: the first amplitude ratio $v_1/r$ vs. the frequency ratio $\omega/\omega_1$ at $x = 0.25l$ in the first span.
In this case, a bifurcation point is also found on the backbone curve. The determinant of the matrix $D$ is equal to zero for a non-dimensional frequency in the range $1.329 < \omega/\omega_n < 1.344$. As in the previous example, the procedure described above is used to determine a post-bifurcation path of the backbone curve. As in example 1, the bifurcation point on a backbone curve exists in a range of internal resonance where commensurable frequencies are associated with the symmetrical and antisymmetrical modes of vibration, respectively. This is an important and probably general necessary condition for existence of bifurcation points on backbone curves. However, this problem requires further and deeper studies. Finally, the first part of the backbone curve discussed is shown in Figures 14 and 15. It is visible that the first loop shown in these figures is similar to one pictured in Figures 7 and 8.
5. CONCLUDING REMARKS

The finite element method together with the continuation method have been used to study non-linear free vibrations of beams in the presence of internal resonance. The complicated backbone curves have been successfully constructed in a fully automatic way. The bifurcation points have also been found on some backbone curves for simple beam structures. Moreover, the post-bifurcation paths of the backbone curves have also been determined by using a rather simple numerical procedure. It was found that bifurcation points on the backbone curve exist if two commensurable frequencies are associated with the symmetrical and antisymmetrical modes of vibration, respectively. The existence of these points on the beams backbone curves have been reported for the first time. Several theoretical and numerical problems connected with bifurcation points require further studies which are in course. The results will be reported in a separate paper.
ACKNOWLEDGMENT

The author wishes to express his gratitude to an anonymous referee for his comments, which resulted in a considerably improved paper.

REFERENCES