

## Sensitivity analysis for structures executing nonlinear free and steady state vibration

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### Abstract

In this paper, the sensitivity analysis for geometrically nonlinear structures executing free and steady state vibration is presented. The simple and direct approach to sensitivity analysis is proposed. In the proposed method the numerical integration of motion equations of the original and adjoint structures are omitted because the sensitivity analysis in the frequency domain is presented. The first-order sensitivity coefficients of structures with respect to material and geometric parameters are determined. Beams with immovable ends are taken into account as an example of geometrically nonlinear structure. The structures are treated as discrete systems for which the motion equations are obtained by the FEM methods.

*Keywords: sensitivity analysis, geometric nonlinearity, nonlinear vibration, free and steady state vibration,*

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### Introduction

An analysis of free and steady state vibration is an important task in designing processes of structures in many branches of engineering. In this context the sensitivity analysis of free and steady state responses are a necessary step for optimum design of vibrating structures. A large amount of literature on sensitivity analysis of linear dynamic systems has been published in recent years. However, most of it is concerned with parameter sensitivity of frequencies and modes of vibration of structures. Among others the sensitivity of frequencies and modes of vibration of linear systems is discussed in [1–5]. Mostly, systems without damping are considered. However, in papers [3–5] the design sensitivity of frequencies and modes of vibration of damped systems are also considered. Only a limited number of papers are available on sensitivity analysis of amplitudes of undamped or damped steady state structural vibration [6–13]. Papers concerning the sensitivity analysis of geometrically nonlinear structures executing free or steady state vibration are extremely rare [14–16]. A recent review of publication concerning the sensitivity analysis is given in paper [17].

In this paper, the sensitivity analysis for geometrically nonlinear structures executing free and steady state vibration is presented. In particular, the nonlinear dynamic systems with cubic nonlinearities are investigated. The beams and plates vibrating with average and large amplitudes are most important examples of such structural systems. The simple and direct approach to sensitivity analysis is proposed. In the proposed method the numerical integration of motion equations of the original and adjoint structures are omitted because the sensitivity analysis in the frequency domain is presented. In particular, systems with viscoelastic properties are considered because these types of structures are rarely investigated but could be important when the passive system of vibration reduction is installed on structure. The first-order sensitivity coefficients of structures with respect to material and geometric parameters are determined.

The main properties of nonlinear systems executing free or steady state vibration is usually presented in compact way on the response curves (i.e. on the backbone curves or on the resonance curves, respectively). These curves show the dependence of system response on some appropriately chosen

systems parameters. In a field of nonlinear dynamics, the amplitude of vibration and the frequency or amplitude of excitation forces are usually chosen main parameters. For this reason, the system sensitivity analysis along the response curves is considered in the paper. This type of analysis isn't presented in previous paper [14–16].

In many cases, the response curves have complicated shapes with the limit, turning and bifurcation points on it. These means that the general sensitivity analysis must take into account these facts. Up to now, the sensitivity analysis along the system response curves with singular points on it was analyzed only in a context of statically loaded structures (see [18–20]). A general and mathematically rigorous design sensitivity formulation was presented in [18] for nonlinear elastic structures reaching critical equilibrium states. Solution of many important problems of sensitivity analysis for nonlinear structures can be found in [19]. In this paper the sensitivity at the bifurcation points existing on the backbone curves is analyzed.

Beams with immovable ends are taken into account as an example of geometrically nonlinear structure. The von Karman theory is used to describe the dynamic behaviour of beams. The structures are treated as discrete systems for which the motion equations are obtained by the FEM methods. The results of example calculation illustrating the proposed method are also presented and briefly discussed.

### Equation of motion and equation of amplitude

#### *Equation of motion*

The total Lagrangian formulation is often used for description of behaviour of structures which undergoes large displacements with small strains and rotations. In this case, the beams and plates internal forces can be written as the cubic functions of displacements. If the finite element method is used, the motion equation of harmonically excited structures can be written in the following form (for details, see [21]):

$$\mathbf{r}(t) = \mathbf{M} \ddot{\mathbf{q}}(t) + \mathbf{C} \dot{\mathbf{q}}(t) + \mathbf{K}_0 \mathbf{q}(t) + \frac{1}{2} \mathbf{K}_2(\mathbf{q}) \mathbf{q}(t) - \mathbf{p}_1 \cos \lambda t - \mathbf{p}_2 \cos \lambda t = \mathbf{0} \quad (1)$$

where  $\mathbf{q}(t)$ ,  $\mathbf{p}_1$ ,  $\mathbf{p}_2$  are vector of nodal displacements and vectors of amplitudes of nodal forces, respectively;  $\mathbf{M}$ ,  $\mathbf{C}$  are

the global mass, and damping matrices. The symbols  $t$  and  $\lambda$  denote time and the excitation frequency. Moreover,  $\mathbf{K}_0$  and  $\mathbf{K}_2(\mathbf{q})$  are the global linear and nonlinear stiffness matrices, respectively. On a finite element level, the above mentioned matrices are defined as (see [22], for details):

$$\mathbf{M}_e = \int_V \mathbf{N}^T m \mathbf{N} dV, \quad \mathbf{K}_{0e} = \int_V \mathbf{B}_0^T \mathbf{E} \mathbf{B}_0 dV, \quad (2)$$

$$\mathbf{K}_{2e} = \int_V \mathbf{B}_1^T(\mathbf{q}_e) \mathbf{E} \mathbf{B}_1(\mathbf{q}_e) dV, \quad (3)$$

where  $\mathbf{E}$  denotes the matrix of elasticity,  $V$  is the volume of finite element,  $\mathbf{N}$ ,  $\mathbf{B}_0$ ,  $\mathbf{B}_1(\mathbf{q}_e)$  are the matrix of shape functions and the linear and nonlinear strain-displacement transformation matrices, respectively. The matrix  $\mathbf{B}_1(\mathbf{q}_e)$  is the matrix of linear and homogenous functions of nodal displacements  $\mathbf{q}_e$ .

In this paper, the beam structures are considered as an example of structures with cubic nonlinearities. The nonlinear equation of motion for beams with immovable ends can be written in the form (see [23] for derivation).

$$\mathbf{r}(t) = \mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}_0\mathbf{q}(t) + \frac{EA}{2L}\mathbf{B}\mathbf{q}(t)\mathbf{q}^T(t)\mathbf{B}\mathbf{q}(t) - \mathbf{p}_1 \cos \lambda t - \mathbf{p}_2 \cos \lambda t = \mathbf{0}, \quad (4)$$

where now

$$\mathbf{K}_2(\mathbf{q}) = \frac{EA}{L}\mathbf{B}\mathbf{q}(t)\mathbf{q}^T(t)\mathbf{B}\mathbf{q}(t). \quad (5)$$

Symbols  $E$ ,  $A$ ,  $L$  denote Young's modulus, the area of beam cross-section and the total beam length, respectively. On an element level the transverse displacement  $w(x, t)$  are written in the form:

$$w(x, t) = \mathbf{N}(x)\mathbf{q}_e(t). \quad (6)$$

The two node element is used and the Hermite polynomials are shape functions. On an elemental level the matrix  $\mathbf{B}$  is defined as:

$$\mathbf{B}_e = \int_0^l \mathbf{N}_{,x}^T(x) \mathbf{N}_{,x}(x) dx, \quad (7)$$

where  $l$  is the finite element length and  $\mathbf{N}_{,x} \equiv d\mathbf{N}/dx$ .

#### Amplitude equation

The dynamic behaviour of nonlinear systems can be very complex in comparison with linear ones. Depending on systems parameters the dynamic response of nonlinear systems loaded by harmonically excited forces could be periodic, almost periodic or chaotic [24]. Besides of main resonances the subharmonic resonances, the superharmonic resonances and the jump phenomena can occur. However, very often the damping forces acting on real structures are able to significantly reduce or completely eliminate the possibility of chaotic and almost periodic responses of considered systems. Moreover, the secondary resonances disappear or they are significantly reduced. For these reasons simply one harmonic solution can accurately describe the steady state response of many real structures executing vibration with moderately large and large amplitudes.

Therefore, it is assumed that the steady state response of harmonically excited structure can be described by

$$\mathbf{q}(t) = \mathbf{a}_1 \cos \lambda t + \mathbf{a}_2 \sin \lambda t. \quad (8)$$

The Ritz method (in the time domain) is used to obtain the so-called amplitude equations with respect to the unknown vectors  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ . This approach leads us to the amplitude equations which are identical with one obtained by the harmonic balance method providing that the steady state solution contains identical harmonics.

From the Ritz conditions

$$\frac{2}{T} \int_0^T \mathbf{r}(t) \cos \lambda t dt = \mathbf{0}, \quad \frac{2}{T} \int_0^T \mathbf{r}(t) \sin \lambda t dt = \mathbf{0}, \quad (9)$$

the following amplitude equations are obtained (see [21, 23] for details):

$$\mathbf{r}_1 = (\mathbf{K}_0 - \lambda^2 \mathbf{M} + \mathbf{H}_{11}(\mathbf{a}))\mathbf{a}_1 + (\lambda \mathbf{C} + \mathbf{H}_{12}(\mathbf{a}))\mathbf{a}_2 - \mathbf{p}_1 = \mathbf{0}, \quad (10)$$

$$\mathbf{r}_2 = (-\lambda \mathbf{C} + \mathbf{H}_{21}(\mathbf{a}))\mathbf{a}_1 + (\mathbf{K}_0 - \lambda^2 \mathbf{M} + \mathbf{H}_{22}(\mathbf{a}))\mathbf{a}_2 - \mathbf{p}_2 = \mathbf{0}, \quad (11)$$

where  $T = 2\pi / \lambda$  is the excitation frequency and

$$\mathbf{H}_{11}(\mathbf{a}) = \frac{3}{8}\mathbf{K}_2(\mathbf{a}_1, \mathbf{a}_1) + \frac{1}{8}\mathbf{K}_2(\mathbf{a}_2, \mathbf{a}_2), \quad (12)$$

$$\mathbf{H}_{12}(\mathbf{a}) = \mathbf{H}_{21}(\mathbf{a}) = \frac{1}{8}\mathbf{K}_2(\mathbf{a}_1, \mathbf{a}_2) + \frac{1}{8}\mathbf{K}_2(\mathbf{a}_2, \mathbf{a}_1), \quad (13)$$

$$\mathbf{H}_{22}(\mathbf{a}) = \frac{1}{8}\mathbf{K}_2(\mathbf{a}_1, \mathbf{a}_1) + \frac{3}{8}\mathbf{K}_2(\mathbf{a}_2, \mathbf{a}_2). \quad (14)$$

In relations (9) symbol  $\mathbf{r}(t)$  denotes the vector of residuals which we obtain introducing the assumed solution (8) into the equation of motion (1). Moreover, on a level of finite element the  $\mathbf{K}_2(\mathbf{a}_1, \mathbf{a}_2)$  matrix is defined as follows (see [21])

$$\mathbf{K}_{2e}(\mathbf{a}_1, \mathbf{a}_2) = \int_V \mathbf{B}_1^T(\mathbf{a}_1) \mathbf{E} \mathbf{B}_1(\mathbf{a}_2) dV. \quad (15)$$

The amplitude equations for beams structures take the form of equations (10) and (11) with nonlinear matrices defined by:

$$\mathbf{H}_{11}(\mathbf{a}) = \left( \frac{3EA}{8L} \mathbf{a}_1^T \mathbf{B} \mathbf{a}_1 + \frac{EA}{8L} \mathbf{a}_2^T \mathbf{B} \mathbf{a}_2 \right) \mathbf{B}, \quad (16)$$

$$\mathbf{H}_{12}(\mathbf{a}) = \mathbf{H}_{21}(\mathbf{a}) = \frac{EA}{4L} \mathbf{a}_1^T \mathbf{B} \mathbf{a}_2 \mathbf{B}, \quad (17)$$

$$\mathbf{H}_{22}(\mathbf{a}) = \left( \frac{3EA}{8L} \mathbf{a}_2^T \mathbf{B} \mathbf{a}_2 + \frac{EA}{8L} \mathbf{a}_1^T \mathbf{B} \mathbf{a}_1 \right) \mathbf{B}. \quad (18)$$

The amplitude equations (10) and (11) could be written as one matrix equation of the form

$$\tilde{\mathbf{r}} = (\tilde{\mathbf{K}} - \lambda^2 \tilde{\mathbf{M}} + \lambda \tilde{\mathbf{C}} + \tilde{\mathbf{H}}(\tilde{\mathbf{a}}))\tilde{\mathbf{a}} - \tilde{\mathbf{p}} = \mathbf{0}, \quad (19)$$

where  $\tilde{\mathbf{r}} = \text{col}(\mathbf{r}_1, \mathbf{r}_2)$ ,  $\tilde{\mathbf{p}} = \text{col}(\mathbf{p}_1, \mathbf{p}_2)$ ,  $\tilde{\mathbf{a}} = \text{col}(\mathbf{a}_1, \mathbf{a}_2)$ ,

$$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \quad \tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{0} & \mathbf{C} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix}, \quad (20)$$

$$\tilde{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_0 \end{bmatrix}, \quad \tilde{\mathbf{H}}(\tilde{\mathbf{a}}) = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix}. \quad (21)$$

For a given excitation frequency  $\lambda$ , from the amplitude equation (19) the unknown vector of amplitudes  $\tilde{\mathbf{a}}$  could be determined. However, as in a case of linear systems, we are interested in determination of the response curve, i.e. dependence of amplitudes of vibration versus the excitation frequency. In this context, the amplitude equation is treated as the equation with parameter. In a given range of excitation frequency, say  $(\lambda_a, \lambda_b)$ , the amplitude equation could be solving using the continuation method. Detailed description of continuation method applied to solve the above mentioned amplitude equation is presented in [21, 23].

Here only the incremental form of amplitude equation together with the tangent matrix  $\tilde{\mathbf{K}}_T(\tilde{\mathbf{a}})$  is written for convenience

$$\tilde{\mathbf{K}}_T(\tilde{\mathbf{a}})\Delta\tilde{\mathbf{a}} = -\tilde{\mathbf{r}}(\tilde{\mathbf{a}}), \quad (22)$$

where

$$\tilde{\mathbf{K}}_T(\tilde{\mathbf{a}}) = \tilde{\mathbf{K}} - \lambda^2\tilde{\mathbf{M}} + \lambda\tilde{\mathbf{C}} + \tilde{\mathbf{Z}}(\tilde{\mathbf{a}}), \quad (23)$$

$$\tilde{\mathbf{Z}}(\tilde{\mathbf{a}}) = \begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix}, \quad (24)$$

$$\mathbf{Z}_{11}(\mathbf{a}) = \frac{3}{4}[\mathbf{K}_2(\mathbf{a}_1, \mathbf{a}_1) + \mathbf{K}_5(\mathbf{a}_1, \mathbf{a}_1)] + \frac{1}{4}[\mathbf{K}_2(\mathbf{a}_2, \mathbf{a}_2) + \mathbf{K}_5(\mathbf{a}_2, \mathbf{a}_2)], \quad (25)$$

$$\mathbf{Z}_{12}(\mathbf{a}) = \mathbf{Z}_{21}(\mathbf{a}) = \frac{1}{4}[\mathbf{K}_2(\mathbf{a}_1, \mathbf{a}_2) + \mathbf{K}_5(\mathbf{a}_1, \mathbf{a}_2)] + \frac{1}{4}[\mathbf{K}_2(\mathbf{a}_2, \mathbf{a}_1) + \mathbf{K}_5(\mathbf{a}_2, \mathbf{a}_1)], \quad (26)$$

$$\mathbf{Z}_{22}(\mathbf{a}) = \frac{1}{4}[\mathbf{K}_2(\mathbf{a}_1, \mathbf{a}_1) + \mathbf{K}_5(\mathbf{a}_1, \mathbf{a}_1)] + \frac{3}{4}[\mathbf{K}_2(\mathbf{a}_2, \mathbf{a}_2) + \mathbf{K}_5(\mathbf{a}_2, \mathbf{a}_2)]. \quad (27)$$

On an element level the matrix  $\mathbf{K}_5(\mathbf{a}_1, \mathbf{a}_2)$  is defined as (see [21], for details)

$$\mathbf{K}_{5e}(\mathbf{a}_1, \mathbf{a}_2) = \int_V \mathbf{G}^T \mathbf{S}_2(\mathbf{a}_1, \mathbf{a}_2) \mathbf{G} dV. \quad (28)$$

For considered beam structures we have

$$\mathbf{Z}_{11}(\mathbf{a}) = \mathbf{H}_{11}(\mathbf{a}) + \frac{EA}{4L} \mathbf{B}(3\mathbf{a}_1\mathbf{a}_1^T + \mathbf{a}_2\mathbf{a}_2^T) \mathbf{B}, \quad (29)$$

$$\mathbf{Z}_{12}(\mathbf{a}) = \mathbf{Z}_{21}(\mathbf{a}) = \mathbf{H}_{12}(\mathbf{a}) + \frac{EA}{4L} \mathbf{B}(\mathbf{a}_1\mathbf{a}_2^T + \mathbf{a}_2\mathbf{a}_1^T) \mathbf{B}, \quad (30)$$

$$\mathbf{Z}_{22}(\mathbf{a}) = \mathbf{H}_{22}(\mathbf{a}) + \frac{EA}{4L} \mathbf{B}(\mathbf{a}_1\mathbf{a}_1^T + 3\mathbf{a}_2\mathbf{a}_2^T) \mathbf{B}. \quad (31)$$

Typical response curves are shown in Figs 1 and 2. In Fig.1 the appropriately scaled typical element of vector  $\mathbf{a}_1$  versus the nondimensional excitation frequency is shown, while in Fig.2 the element of vector  $\mathbf{a}_2$  is shown in a similar way. Please notice, the limit and turning points, marked as small circles, exist on these curves.

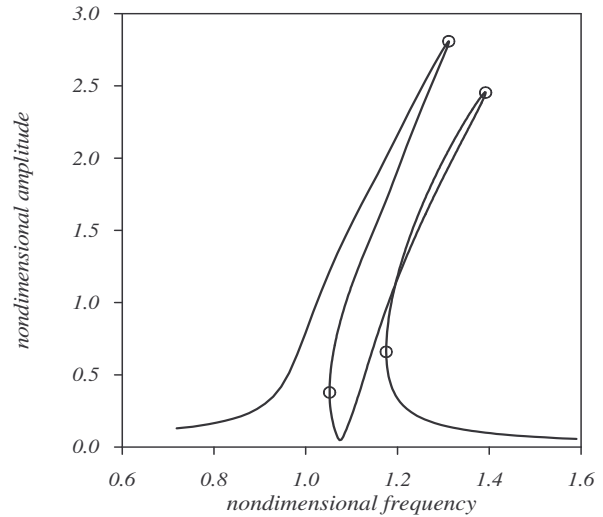


Fig.1 Example of response curves with critical points

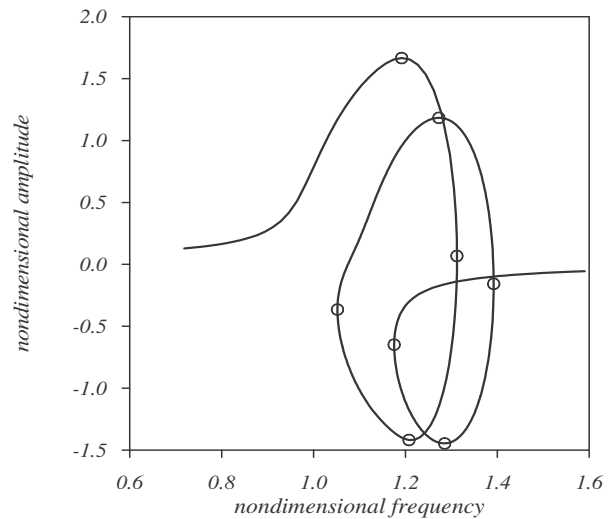


Fig. 2 Example of response curve with critical points

In a case of free vibration:  $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{0}$ ,  $\mathbf{C} = \mathbf{0}$  and the solution of motion equation (1) is assumed in the form:

$$\mathbf{q}(t) = \mathbf{a} \cos \omega t, \quad (32)$$

where  $\omega$  is the nonlinear natural frequency.

The amplitude equation which is now obtained from the following Ritz condition

$$\frac{2}{T} \int_0^T \mathbf{r}(t) \cos \omega t dt = \mathbf{0}, \quad (33)$$

takes the form ( $\lambda = \omega^2$ ):

$$(\mathbf{K}_0 - \lambda \mathbf{M} + \frac{3}{8} \mathbf{K}_2(\mathbf{a}, \mathbf{a})) \mathbf{a} = \mathbf{0}. \quad (34)$$

For beams structures we can write

$$\mathbf{K}_2(\mathbf{a}) = \frac{EA}{L} \mathbf{B} \mathbf{a} \mathbf{a}^T \mathbf{B} \equiv \frac{EA}{L} \mathbf{a}^T \mathbf{B} \mathbf{a} \mathbf{B}. \quad (35)$$

Equation (34) can be understood as the nonlinear eigenvalue problem because now both  $\lambda$  and  $\mathbf{a}$  are unknown. This equation could be also treated as the matrix equation with parameter and can be solve using the continuation method as it is described in [21]. The natural frequency  $\omega$  or the appropriately chosen element of vector  $\mathbf{a}$  is usually chosen as the main parameter.

The considered equation has the trivial solution:  $\mathbf{a} = \mathbf{0}$  for all  $\omega$ . In the  $(\mathbf{a}, \omega)$  space the trivial solution is a line coinciding with  $\omega$ - axis, as it is shown in Fig.3. On this line the bifurcation points, visible as small circles, exist. The bifurcation point has coordinates:  $\mathbf{a} = \mathbf{0}$  and  $\omega = \omega_l$ , where  $\omega_l$  is simply the linear frequency of vibration. The nontrivial solutions of equation (34) also exist and are known as the backbone curves if they are presented graphically. The backbone curves emanate from the previously mentioned bifurcation points. For systems with cubic nonlinearities the typical backbone curve has a shape shown in Fig.3. Please notice, except mentioned above, there are no critical points on the backbone curve. However, it is true only for systems with cubic nonlinearities and when the one harmonic solution of motion equation is valid.

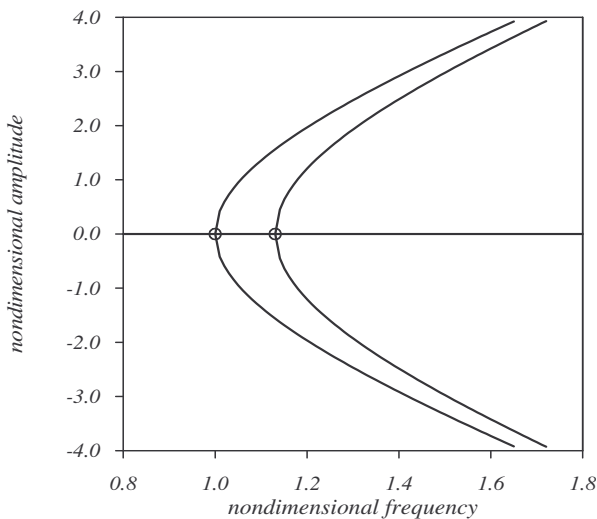


Fig.3 Typical example of backbone curves

In a case of free vibration, the tangent matrix  $\mathbf{K}_T(\mathbf{a})$  related with the amplitude equation (34) is given by

$$\mathbf{K}_T(\mathbf{a}) = \mathbf{K}_0 - \lambda \mathbf{M} + \frac{3}{4} [\mathbf{K}_2(\mathbf{a}, \mathbf{a}) + \mathbf{K}_5(\mathbf{a}, \mathbf{a})], \quad (36)$$

$$\mathbf{K}_T(\mathbf{a}) = \mathbf{K}_0 - \lambda \mathbf{M} + \frac{3EA}{8L} (\mathbf{a}^T \mathbf{B} \mathbf{a} \mathbf{B} + 2 \mathbf{B} \mathbf{a} \mathbf{a}^T \mathbf{B}), \quad (37)$$

for general systems with cubic nonlinearities and for beam structures, respectively.

Sometimes it is useful to introduce the normalized nonlinear eigenvector  $\bar{\mathbf{a}}$  in such a way that

$$\mathbf{a} = \alpha \bar{\mathbf{a}}, \quad (38)$$

where  $\alpha$  is the scaling parameter.

If, for example  $\alpha$  is the amplitude of vibration at the particular point of system, say point  $p$ , then  $\bar{a}_p = 1$ , and changes of elements of the  $\bar{\mathbf{a}}$  vector represents relative

changes of amplitudes of vibration at different points with respect to amplitude of vibration at point  $p$ .

The second choice of  $\alpha$  could be  $\alpha^2 = \tilde{m}$ , where  $\tilde{m} = \mathbf{a}^T \mathbf{M} \mathbf{a}$ . In this case, the  $\bar{\mathbf{a}}$  vector is normalized in such a way that  $\bar{\mathbf{a}}^T \mathbf{M} \bar{\mathbf{a}} = 1$  for all  $\alpha$ .

After introducing (38) into (34), the amplitude equation could be rewritten in the form

$$\left( \mathbf{K}_0 - \lambda \mathbf{M} + \frac{3}{8} \alpha^2 \mathbf{K}_2(\bar{\mathbf{a}}, \bar{\mathbf{a}}) \right) \bar{\mathbf{a}} = \mathbf{0}. \quad (39)$$

In the above approach the nonlinearity of the system is mainly governed by the introduced parameter  $\alpha$ . If value of  $\alpha$  goes to zero the system behaviour is essentially linear, the third term in equation (39) can be neglected as small in comparison with first and second one and the amplitude equation (39) transform to the linear eigenvalue problem. Please notice, the  $\bar{\mathbf{a}}$  vector do not vanish and smoothly transform to the linear eigenvector.

#### Conditions of critical points appearing on the response curve

The conditions of critical points existence will be briefly discussed because the tangent matrix isn't symmetric in a case of steady state forced vibration. Differentiating (19) one obtains the following equation

$$\tilde{\mathbf{K}}_T(\tilde{\mathbf{a}}, \lambda) d\tilde{\mathbf{a}} + \tilde{\mathbf{r}}_\lambda(\tilde{\mathbf{a}}, \lambda) d\lambda = \mathbf{0}, \quad (40)$$

which must be satisfied on the response curve. Here,  $\tilde{\mathbf{r}}_\lambda(\tilde{\mathbf{a}}, \lambda) = \partial \tilde{\mathbf{r}} / \partial \lambda$ .

Equation (40) could be solved with respect to  $d\tilde{\mathbf{a}}$  as long as the matrix  $\tilde{\mathbf{K}}_T(\tilde{\mathbf{a}}, \lambda)$  isn't singular. It can be checked solving one from the following linear eigenvalue problems:

$$\tilde{\mathbf{K}}_T(\tilde{\mathbf{a}}, \lambda) \mathbf{u} = \mu \mathbf{u}, \quad \mathbf{v}^T \tilde{\mathbf{K}}_T(\tilde{\mathbf{a}}, \lambda) = \mu \mathbf{v}^T, \quad (41)$$

where  $\mathbf{v}$  and  $\mathbf{u}$  are the left and right eigenvectors, respectively and  $\mu$  is the eigenvalue.

The matrix  $\tilde{\mathbf{K}}_T(\tilde{\mathbf{a}}, \lambda)$  is singular if for  $\tilde{\mathbf{a}} = \tilde{\mathbf{a}}_c$  at least one eigenvalue, say  $\mu_1$ , is equal zero. The corresponding eigenvector will be denoted as  $\mathbf{v}_c$  and  $\mathbf{u}_c$ .

Pre-multiplying equation (40) by  $\mathbf{v}_c^T$  and taking into account that at the critical point

$$\mathbf{v}_c^T \tilde{\mathbf{K}}_T(\tilde{\mathbf{a}}_c, \lambda_c) = 0, \quad (42)$$

we obtain

$$\mathbf{v}_c^T \tilde{\mathbf{r}}_\lambda(\tilde{\mathbf{a}}_c, \lambda) d\lambda = 0, \quad (43)$$

what means that

$$d\lambda = 0, \quad (44)$$

at the limit point and

$$\mathbf{v}_c^T \tilde{\mathbf{K}}_T(\tilde{\mathbf{a}}_c, \lambda) = 0, \quad (45)$$

at the bifurcation point.

In a case of free vibration  $\mathbf{v} = \mathbf{u}$  because the corresponding tangent matrix is symmetric.

**Sensitivity analysis**

*Sensitivity of frequency and mode of vibration at regular states*

First of all the sensitivity of frequency and mode of vibration to variation of design parameter  $s$  will be considered using the method described in [2] for linear systems.

The nonlinear eigenvalue problem (34) consists of  $n+1$  unknowns, i.e.  $\lambda$  and  $\mathbf{a}$ . An additional equation, which can be understood as the condition of normalization of  $\mathbf{a}$ , is chosen in the form:

$$\mathbf{a}^T \mathbf{M} \mathbf{a} = \alpha^2, \quad (46)$$

where  $\alpha$  is the fixed parameter. The value of parameter  $\alpha$  can be easily calculated for a given solution of equation (34). The second choice of above mentioned normalization condition, which can be interpreted as the maximal kinetic energy of the dynamic system, could be in the form:

$$\frac{1}{2} \lambda \mathbf{a}^T \mathbf{M} \mathbf{a} = \alpha^2. \quad (47)$$

Now equations (34) and (46) or (47) are treated as a system of equations.

In generality, except of  $\alpha$ , each term of equations (34) and (47) can depends on parameter  $s$ . Taking the first derivative of mentioned above equations with respect to design parameter  $s$  and using the symbol like  $\partial \mathbf{M} / \partial s$  to denote the explicit derivatives of  $\mathbf{M}$  with respect to  $s$  we obtain the following set of linear equations:

$$\mathbf{K}_T(\mathbf{a}) \frac{d\mathbf{a}}{ds} - \mathbf{M} \mathbf{a} \frac{d\lambda}{ds} = - \left( \frac{\partial \mathbf{K}_0}{\partial s} - \lambda \frac{\partial \mathbf{M}}{\partial s} + \frac{3}{8} \frac{\partial \mathbf{K}_2}{\partial s} \right) \mathbf{a}, \quad (48)$$

$$-\mathbf{a}^T \mathbf{M} \frac{d\mathbf{a}}{ds} - \frac{1}{2\lambda} \mathbf{a}^T \mathbf{M} \frac{d\lambda}{ds} = \frac{1}{2} \lambda \mathbf{a}^T \frac{\partial \mathbf{M}}{\partial s} \mathbf{a}, \quad (49)$$

from which the searched derivatives  $d\mathbf{a}/ds$  and  $d\lambda/ds$  could be calculated. If the normalization condition has the form (46) we obtain, instead of equation (49), the following one:

$$-\mathbf{a}^T \mathbf{M} \frac{d\mathbf{a}}{ds} = \frac{1}{2} \mathbf{a}^T \frac{\partial \mathbf{M}}{\partial s} \mathbf{a}. \quad (50)$$

The matrices of explicit derivatives with respect to  $s$  i.e.  $\partial \mathbf{M} / \partial s$ ,  $\partial \mathbf{K}_0 / \partial s$  and  $\partial \mathbf{K}_2 / \partial s$  can be relatively easy calculated in a finite element level and assembled in an usual way.

*Sensitivity of frequency and mode of vibration at the bifurcation point*

As it was mentioned above, the bifurcation point has coordinates  $\mathbf{a}_c = \mathbf{a}_b = \mathbf{0}$ ,  $\lambda = \lambda_b = \omega_l^2$ , where  $\omega_l$  is the chosen natural frequency of linear system. At the bifurcation point  $\mathbf{K}_T(\mathbf{a}_b) = \mathbf{K}_0 - \lambda_b \mathbf{M}$  and the equation (48) could be rewritten in the form:

$$(\mathbf{K}_0 - \lambda_b \mathbf{M}) \frac{d\mathbf{a}_b}{ds} = \mathbf{0}. \quad (51)$$

Because  $\det(\mathbf{K}_0 - \lambda_b \mathbf{M}) = 0$  and the mode of vibration  $\mathbf{v}_l$  of the corresponding linear system fulfils the equation

$$(\mathbf{K}_0 - \lambda_b \mathbf{M}) \mathbf{v}_l = \mathbf{0}, \quad (52)$$

the sensitivity of  $\mathbf{a}$  with respect to  $s$  at the bifurcation point is proportional to the linear mode of vibration, i.e.

$$\frac{d\mathbf{a}_b}{ds} = \varepsilon \mathbf{v}_l, \quad (53)$$

where  $\varepsilon$  is a free parameter.

Please notice, it is true for all design parameters.

Consider now, the amplitude equation (34) which could be rewritten in the slightly different form

$$\mathbf{K}_s(\mathbf{a}, \lambda) \mathbf{a} = \mathbf{0}, \quad (54)$$

where  $\mathbf{K}_s(\mathbf{a}, \lambda)$  is the secant matrix defined as

$$\mathbf{K}_s(\mathbf{a}, \lambda) = \mathbf{K}_0 - \lambda \mathbf{M} + \frac{3}{8} \mathbf{K}_2(\mathbf{a}, \mathbf{a}). \quad (55)$$

The condition of existence of the nontrivial solution can be now written in too equivalent forms:

$$\det \mathbf{K}_s(\mathbf{a}_b, \lambda_b) = 0, \quad (56)$$

$$\mathbf{z} = \mathbf{K}_s(\mathbf{a}_b, \lambda_b) \mathbf{v} = \mathbf{0}. \quad (57)$$

The  $\mathbf{v}$  vector appearing in equation (57) is the solution of the following linear eigenvalue problem

$$\mathbf{K}_s(\mathbf{a}_b, \lambda_b) \mathbf{v} = \mu \mathbf{v}, \quad (58)$$

corresponding to  $\mu = 0$ .

Suppose, as happens in considered case, that  $\mathbf{a}_b = \mathbf{0}$ . Now, the equation (57) takes the form:

$$(\mathbf{K}_0 - \lambda_b \mathbf{M}) \mathbf{v} = \mathbf{0}, \quad (59)$$

what means that  $\mathbf{v} = \mathbf{v}_l$  because eigenvalue problems (58) and (52) have identical form.

Differentiating equation (57) with respect to  $s$  and taking into account that the  $\mathbf{v}_l$  vector depends on  $s$  also, we obtain

$$\begin{aligned} \frac{\partial \mathbf{z}}{\partial \mathbf{a}_b} \frac{d\mathbf{a}_b}{ds} + \frac{\partial \mathbf{z}}{\partial \lambda_b} \frac{d\lambda_b}{ds} + \frac{\partial \mathbf{z}}{\partial \mathbf{v}_l} \frac{d\mathbf{v}_l}{ds} + \\ + \left( \frac{\partial \mathbf{K}_0}{\partial s} - \lambda_b \frac{\partial \mathbf{M}}{\partial s} + \frac{3}{8} \frac{\partial \mathbf{K}_2}{\partial s} \right) \mathbf{v}_l = \mathbf{0}, \end{aligned} \quad (60)$$

where

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}_b} = \frac{3}{8} \frac{\partial \mathbf{K}_2}{\partial \mathbf{a}_b} \mathbf{v}_l, \quad \frac{\partial \mathbf{z}}{\partial \lambda_b} = -\mathbf{M} \mathbf{v}_l, \quad (61)$$

$$\frac{\partial \mathbf{z}}{\partial \mathbf{v}_l} = \mathbf{K}_s(\mathbf{a}_b, \lambda). \quad (62)$$

For  $\mathbf{a}_b = \mathbf{0}$ ,

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}_b} = \mathbf{K}_0 - \lambda_b \mathbf{M}, \quad \frac{\partial \mathbf{K}_2}{\partial \mathbf{a}_b} = \mathbf{0}, \quad (63)$$

because the  $\mathbf{K}_2(\mathbf{a})$  matrix is homogenous and quadratic function of  $\mathbf{a}$ .

Introducing relations (61) – (62) into equation (60) we can rewrite it in the form:

$$(\mathbf{K}_0 - \lambda_b \mathbf{M}) \frac{d\mathbf{v}_l}{ds} - \mathbf{M} \mathbf{v}_l \frac{d\lambda_b}{ds} - \left( \frac{\partial \mathbf{K}_0}{\partial s} - \lambda_b \frac{\partial \mathbf{M}}{\partial s} \right) \mathbf{v}_l = \mathbf{0} . \quad (64)$$

Pre-multiplying equation (64) by  $\mathbf{v}_l^T$ , taking into account the symmetry of matrices  $\mathbf{M}$  and  $\mathbf{K}_0$  and using equation (52) the following equation is obtained

$$\frac{d\lambda_b}{ds} = - \frac{\mathbf{v}_l^T \left( \frac{\partial \mathbf{K}_0}{\partial s} - \lambda_b \frac{\partial \mathbf{M}}{\partial s} \right) \mathbf{v}_l}{\mathbf{v}_l^T \mathbf{M} \mathbf{v}_l} . \quad (65)$$

Please notice, the sensitivity of eigenvalue at the bifurcation point is equal the sensitivity of eigenvalue (the square of natural frequency) of the linear system. Moreover, the sensitivity of the  $\mathbf{a}$  vector could be determined with accuracy to constant.

*Design sensitivity of amplitudes of steady state vibration at regular states*

It is assumed, the structural response functional depends on the vector  $\tilde{\mathbf{a}}$  and parameter  $s$ , i.e.  $g(\tilde{\mathbf{a}}(s), s)$ . For instance, we can take  $g(\tilde{\mathbf{a}}(s), s) = \tilde{\mathbf{a}}^T(s) \tilde{\mathbf{a}}(s)$ . Sensitivity gradient of the response functional with respect to  $s$  takes the form:

$$\frac{dg}{ds} = \frac{\partial g}{\partial s} + \frac{\partial g}{\partial \tilde{\mathbf{a}}} \frac{d\tilde{\mathbf{a}}}{ds} . \quad (66)$$

The vector of derivatives of  $\tilde{\mathbf{a}}$  with respect to  $s$  could be determined from the amplitude equation. Differentiating equation (19) with respect to  $s$  and using the chain rule of differentiation we have

$$\frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{\mathbf{a}}} \frac{d\tilde{\mathbf{a}}}{ds} + \frac{\partial \tilde{\mathbf{r}}}{\partial s} = \mathbf{0} . \quad (67)$$

A part of derivatives of  $\tilde{\mathbf{r}}$  which depends explicitly on  $s$  could be written in the form

$$-\frac{\partial \tilde{\mathbf{r}}}{\partial s} = \tilde{\mathbf{f}}(\mathbf{a}, \lambda) = \frac{d\tilde{\mathbf{p}}}{ds} - \left( \frac{\partial \tilde{\mathbf{K}}}{\partial s} - 2\lambda \frac{\partial \lambda}{\partial s} \tilde{\mathbf{M}} - \lambda^2 \frac{\partial \tilde{\mathbf{M}}}{\partial s} + \frac{\partial \lambda}{\partial s} \tilde{\mathbf{C}} + \lambda \frac{\partial \tilde{\mathbf{C}}}{\partial s} \right) \tilde{\mathbf{a}} . \quad (68)$$

Moreover,

$$\frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{\mathbf{a}}} = \mathbf{K}_T(\tilde{\mathbf{a}}, \lambda) , \quad (69)$$

and finally, the vector  $d\tilde{\mathbf{a}}/ds$  could be determined from

$$\mathbf{K}_T(\tilde{\mathbf{a}}, \lambda) \frac{d\tilde{\mathbf{a}}}{ds} = \tilde{\mathbf{f}}(\tilde{\mathbf{a}}, \lambda) . \quad (70)$$

**Results of example calculation**

*Simple analytical solution*

Simple analytical solution exists for the simply supported beam. The equation of motion of beam treated as a continuous system can be written in the form (see [25]):

$$m\ddot{w}(x, t) + EJw_{,xxxx}(x, t) - \frac{EA}{L} \int_0^L w_{,x}^2 dx w_{,xx}(x, t) = 0 , \quad (71)$$

where  $(\circ)_{,x} = d(\circ)/dx$ , symbols  $m$ ,  $EJ$  and  $w$  denote the mass per unit length of beam, the beam rigidity and the transverse displacements of beam, respectively.

If the solution of motion equation has been assumed in the form:

$$w(x, t) = a \sin \frac{\pi x}{l} \cos \omega t , \quad (72)$$

then the amplitude equation is (see [25], for details)

$$\left[ 1 - \left( \frac{\omega}{\omega_l} \right)^2 + \frac{3}{16} \left( \frac{a}{i} \right)^2 \right] a = 0 , \quad (73)$$

where  $\omega_l$  is the fundamental, linear frequency of vibration,  $i$  denotes the radius of inertia of the cross-section.

It is easy to notice, the following function describe the backbone curve

$$\left( \frac{\omega}{\omega_l} \right)^2 = 1 + \frac{3}{16} \left( \frac{a}{i} \right)^2 . \quad (74)$$

For beams with the rectangular cross-section of height  $h$  and width  $b$

$$i^2 = h^2 / 12 , \quad J = bh^3 / 12 , \quad m = \rho bh , \quad (75)$$

$$\omega_l^2 = (\pi / L)^4 EJ / m = (\pi / L)^4 Eh^2 / (12\rho) , \quad (76)$$

where  $\rho$  is the mass density per unit volume.

The height of beam is chosen as the design parameter, i.e.  $s \equiv h$ . The heights of cross-section after and before perturbation are denoted as  $\tilde{h}$  and  $h$ , respectively, i.e.  $\tilde{h} = h + \Delta h$ , where  $\Delta h$  is the design perturbation of height.

The exact nondimensional measure of sensitivity of square of frequency with respect to  $h$  is defined as  $(\omega^2(\tilde{h}) - \omega^2(h)) / \omega_l^2(h)$ . Using relations (74) – (76) and assuming that the amplitude of vibration is identical in both cases we obtain

$$\delta\omega_{exact}^2 = \frac{\omega^2(\tilde{h}) - \omega^2(h)}{\omega_l^2(h)} = \frac{2\Delta h}{h} + \left( \frac{\Delta h}{h} \right)^2 . \quad (77)$$

If nondimensional change of frequency is a measure of sensitivity then

$$\delta\omega_{exact} = \frac{\omega(\tilde{h}) - \omega(h)}{\omega_1(h)} = \frac{\tilde{h}}{h} \sqrt{1 + \frac{9}{4} \left(\frac{a}{h}\right)^2 \left(\frac{h}{\tilde{h}}\right)^2} - \sqrt{1 + \frac{9}{4} \left(\frac{a}{h}\right)^2} . \quad (78)$$

Using the theory of sensitivity we can write

$$\omega^2(\tilde{h}) - \omega^2(h) = \frac{d\omega^2}{dh} \Delta h , \quad (79)$$

$$\frac{d\omega^2}{dh} = \frac{d\omega_1^2}{dh} \left[ 1 + \frac{9}{4} \left(\frac{a}{h}\right)^2 \right] + \frac{9\omega_1^2}{2} \left(\frac{a}{h^3}\right) \left(\frac{da}{dh} h - a\right) , \quad (80)$$

Taking into account that, as previously,

$$\frac{da}{dh} = 0 , \quad \frac{d\omega_1^2}{dh} = \frac{2\omega_1^2}{h} , \quad (81)$$

and after introducing relations (80) and (81) into (79) we obtain

$$\delta\omega_{approx}^2 = \frac{\omega^2(\tilde{h}) - \omega^2(h)}{\omega_1^2(h)} = \frac{2\Delta h}{h} . \quad (82)$$

Proceeding in a similar way we can derive the following formula

$$\delta\omega_{approx} = \frac{\omega(\tilde{h}) - \omega(h)}{\omega_1(h)} = \frac{\Delta h}{h} \frac{1}{\sqrt{1 + \frac{9}{4} \left(\frac{a}{h}\right)^2}} . \quad (83)$$

The formula (82) predict the sensitivity of square of the nonlinear frequency with nondimensional error

$$\mathcal{E} = \delta\omega_{exact}^2 - \delta\omega_{approx}^2 = \left(\frac{\Delta h}{h}\right)^2 , \quad (84)$$

which is very small and independent on the nondimensional amplitude of vibration.

Figure 4 shows the nondimensional change of frequency  $\delta\omega_{exact}$  (the solid line) and the nondimensional change of frequency  $\delta\omega_{approx}$  (the small crosses) resulting from the sensitivity analysis versus the nondimensional height of beam and for different ratios of amplitudes  $a/h$ . The differences of results obtained are also very small as it is shown on Fig. 5.

#### Sensitivity analysis of two degree of freedom system

The free vibration problem of the dynamical system pictured in Fig. 6 is considered. The system of motion equations can be written in the form:

$$\begin{aligned} M_1 \ddot{q}_1 + (k_1 + \mu_1 q_1^2) q_1 - [k_2 + \mu_2 (q_2 - q_1)^2] (q_2 - q_1) &= 0 , \\ M_2 \ddot{q}_2 + [k_2 + \mu_2 (q_2 - q_1)^2] (q_2 - q_1) &= 0 , \end{aligned} \quad (85)$$

Assuming that the system oscillates periodically with one mode we can write

$$q_1 = \alpha \bar{a}_1 \cos \omega t , \quad q_2 = \alpha \bar{a}_2 \cos \omega t , \quad (86)$$

where  $\alpha$  is the amplitude of the second mass, i.e.  $\bar{a}_2 = 1$ .

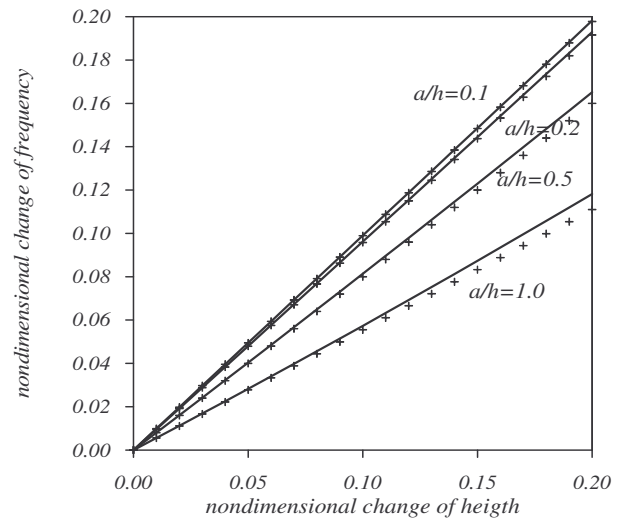


Fig.4 Nondimensional change of frequency versus change of the nondimensional beam height  $\Delta h / h$

Substituting (86) into (85) and applying the Ritz method the following amplitude equation is obtained:

$$\begin{aligned} r_1 &= -\lambda M_1 \bar{a}_1 + \left(k_1 + \frac{3}{4} \alpha^2 \mu_1 \bar{a}_1^2\right) \bar{a}_1 - \left[k_2 + \frac{3}{4} \alpha^2 \mu_2 (\bar{a}_2 - \bar{a}_1)^2\right] (\bar{a}_2 - \bar{a}_1) = 0 , \\ r_2 &= -\lambda M_2 \bar{a}_2 + \left[k_2 + \frac{3}{4} \alpha^2 \mu_2 (\bar{a}_2 - \bar{a}_1)^2\right] (\bar{a}_2 - \bar{a}_1) = 0 , \end{aligned} \quad (87)$$

where  $\lambda = \omega^2$ .

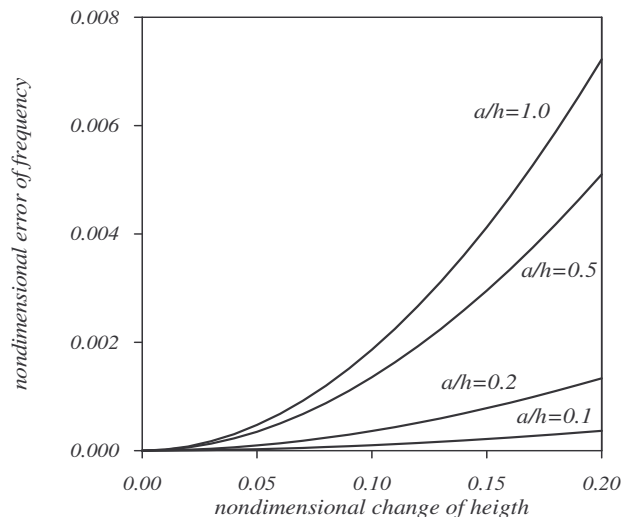


Fig.5 Nondimensional error of frequency versus change of the nondimensional height of beam  $\Delta h / h$

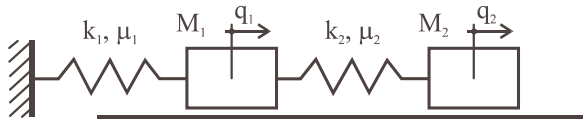


Fig.6 The two degree of freedom system

The sensitivity equations are

$$\begin{aligned} & \left[ k_1 + k_2 + \frac{9}{4} \alpha^2 \mu_2 (\bar{a}_2 - \bar{a}_1)^2 + \frac{9}{4} \alpha^2 \mu_1 \bar{a}_1^2 - \lambda M_1 \right] \frac{d\bar{a}_1}{ds} - \\ & \left[ k_2 + \frac{9}{4} \alpha^2 \mu_2 (\bar{a}_2 - \bar{a}_1)^2 \right] \frac{d\bar{a}_2}{ds} - M_1 \bar{a}_1 \frac{d\lambda}{ds} = -\frac{\partial r_1}{\partial s} , \\ & - \left[ k_2 + \frac{9}{4} \alpha^2 \mu_2 (\bar{a}_2 - \bar{a}_1)^2 \right] \frac{d\bar{a}_1}{ds} + \\ & \left[ k_2 + \frac{9}{4} \alpha^2 \mu_2 (\bar{a}_2 - \bar{a}_1)^2 - \lambda M_2 \right] \frac{d\bar{a}_2}{ds} - M_2 \bar{a}_2 \frac{d\lambda}{ds} = -\frac{\partial r_2}{\partial s} \end{aligned} \quad (88)$$

where symbols  $\partial r_1 / \partial s$  and  $\partial r_2 / \partial s$  are the explicit derivatives of  $r_1$  and  $r_2$  with respect to  $s$ .

In this case the constrains equation is chosen in the form:

$$\bar{a}_2^2 = 1 , \quad (89)$$

what means that

$$\frac{d\bar{a}_2}{ds} = 0 , \quad (90)$$

and during the sensitivity analysis we compare two systems which amplitudes of vibration of the second mass are equal.

Now, the system of sensitivity equations can be rewritten in the form:

$$\begin{aligned} & \left[ k_1 + k_2 + \frac{9}{4} \alpha^2 \mu_2 (\bar{a}_2 - \bar{a}_1)^2 + \frac{9}{4} \alpha^2 \mu_1 \bar{a}_1^2 - \lambda M_1 \right] \frac{d\bar{a}_1}{ds} - \\ & M_1 \bar{a}_1 \frac{d\lambda}{ds} = -\frac{\partial r_1}{\partial s} \\ & - \left[ k_2 + \frac{9}{4} \alpha^2 \mu_2 (\bar{a}_2 - \bar{a}_1)^2 \right] \frac{d\bar{a}_1}{ds} - M_2 \bar{a}_2 \frac{d\lambda}{ds} = -\frac{\partial r_2}{\partial s} . \end{aligned} \quad (91)$$

If we assume that  $\mu_1 = \mu_2 = \mu$  and the parameter  $\mu$  will be chosen as the design ones then

$$\frac{\partial r_1}{\partial \mu} = \frac{3}{4} \alpha^2 \bar{a}_1^3 - \frac{3}{4} \alpha^2 (\bar{a}_2 - \bar{a}_1)^3 , \quad (92)$$

$$\frac{\partial r_2}{\partial \mu} = \frac{3}{4} \alpha^2 (\bar{a}_2 - \bar{a}_1)^3 . \quad (93)$$

The following data are adopted:  $M_1 = M_2 = 10.0kg$  ,  $k_1 = k_2 = 1000.0 N / m$  , and  $\mu = 80000.0 N / m^3$  . The backbone curves  $\omega_1(\alpha)$  is calculated and shown on Fig.7. Please notice, the considered system is strongly nonlinear.

The sensitivity of the first natural frequency  $\omega_1$  and the normalized amplitude of vibration  $\bar{a}_1$  with respect to  $\mu$  are

calculated. It means that nonlinear stiffness in both springs is changed simultaneously. Results are shown on Figs 8 and 9, where mentioned above sensitivities versus the nondimensional frequency  $\omega_1(\alpha) / \omega_{1,lin}$  are presented.

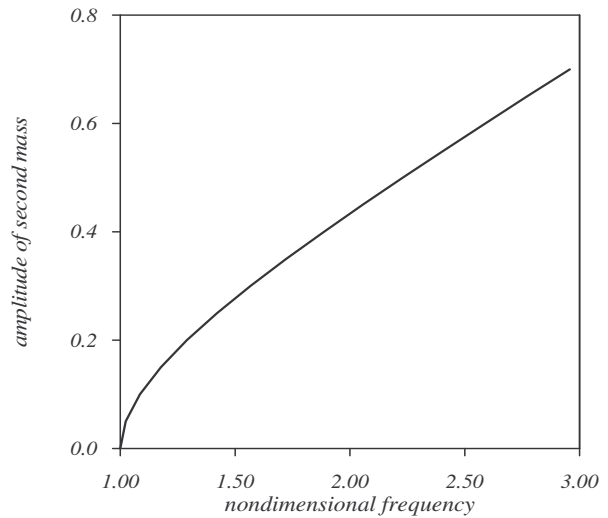


Fig.7 Two mass system – the backbone curve of fundamental mode of vibration

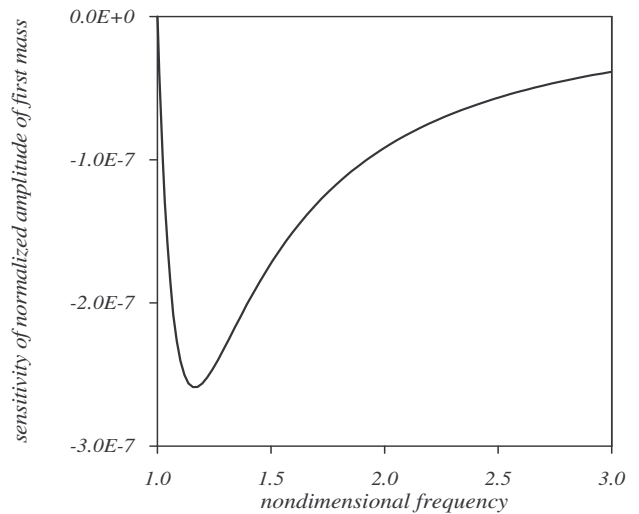


Fig.8 Two mass system – the sensitivity of normalized amplitude of second mass to variation of  $\mu$  versus nondimensional frequency

The accuracy of sensitivity analysis is shown on Figs. 10 and 11. On Fig. 10 we can observe how  $\bar{a}_1$  changes with respect to the amplitude of vibration  $\alpha$  when the nonlinear stiffness of perturbed system has value  $\tilde{\mu} = 88000.0 N / m^3$  , i.e. the nonlinear stiffness of both springs is 10% larger then the stiffness of unperturbed system. Results obtained using the sensitivity analysis is presented as small crosses while the exact results are shown as the solid line. On Fig. 11 similar comparison is made for the nonlinear frequency of vibration. In conclusion we can say, in the considered case the sensitivity analysis give us a good estimation of the perturbed system behaviour.

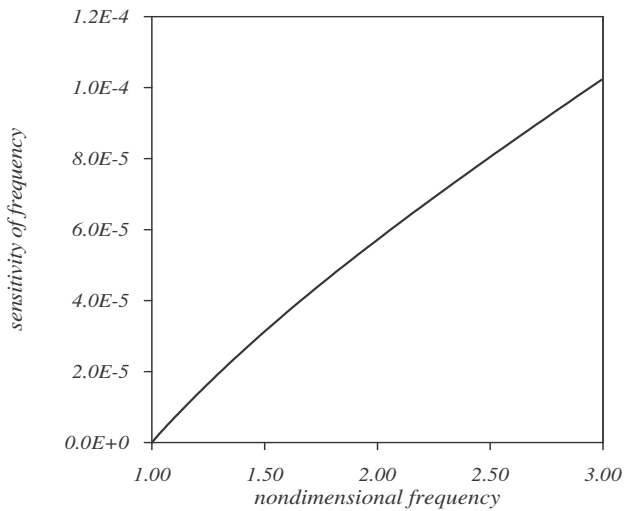


Fig.9 Two mass system – the sensitivity of frequency of vibration to variation of  $\mu$  versus nondimensional frequency

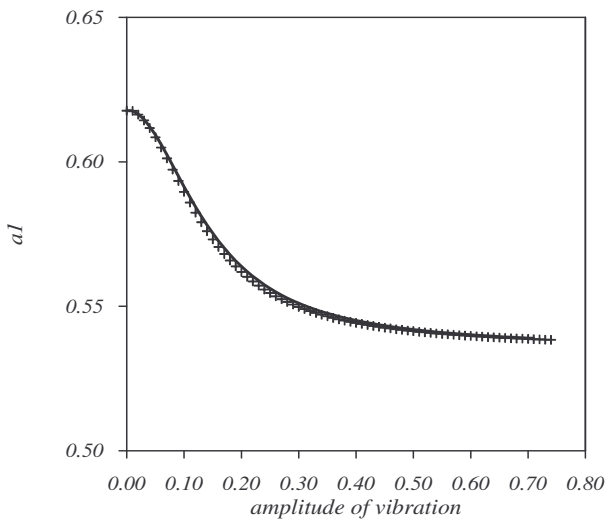


Fig.10 Two mass system – change of  $\bar{a}_1$  with respect to the amplitude of vibration  $\alpha$

*Three span beam - sensitivity of natural frequency and mode of vibration*

Three span beam shown on Fig. 12 are considered. The beam is divided into twenty finite elements. The first and second nonlinear frequencies depend on amplitude measured in the middle of first span as it is shown on Fig. 3. On this figure the nondimensional amplitude is defined as  $a/i$ , where  $i$  is the radius of inertia of the beam cross-section, and the nondimensional frequency is defined as  $\omega/\omega_{lin}$ , where  $\omega_{lin}$  is the fundamental frequency of beam treated as the linear system.

The sensitivity of first nonlinear frequency and mode of vibration with respect to change of the stiffness of elastic support  $k_s$  is calculated. The elastic support is located in the middle of third span. It is assumed that in the initial state  $k_s = 0$  i.e. the elastic support doesn't exist. The results of calculation are shown on Figs. 13 and 14 where the sensitivity

of frequency of vibration and the amplitude of vibration to variation of the stiffness of elastic support are presented.

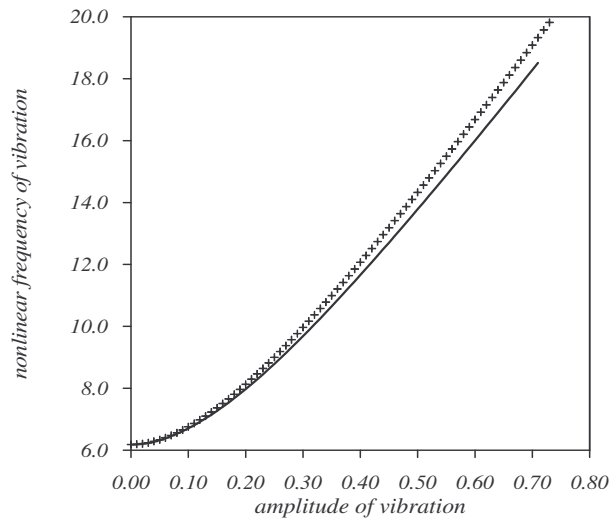


Fig.11 Two mass system – change of frequency of vibration with respect to the amplitude of vibration  $\alpha$

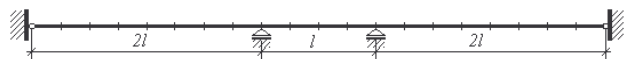


Fig.12 Three span beam

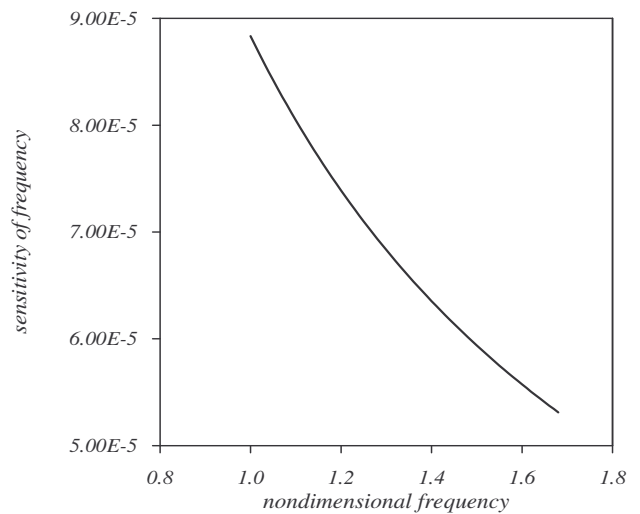


Fig.13 Three span beam – sensitivity of frequency of vibration to variation of stiffness of elastic support versus nondimensional frequency

**Concluding remarks**

In the paper, the design sensitivity of structures vibrating with large amplitudes is considered. Geometrical nonlinearity of systems with so-called cubic nonlinearity is taken into account. The first order sensitivity analysis in the frequency domain is presented. In this way the numerical integration of motion equations of the original and adjoint structures are omitted. The results of example calculation have shown the accuracy and effectiveness of the proposed method.

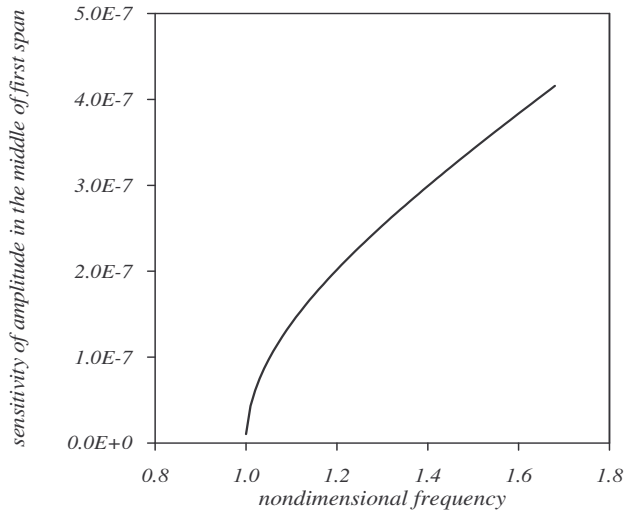


Fig.14 Three span beam – sensitivity of amplitude of vibration in the middle of first span to variation of stiffness of elastic support versus nondimensional frequency of vibration

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