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NON-LINEAR, STEADY STATE VIBRATION OF FRAMES WITH CLEARENCES AT SUPPORTS

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Abstract. *The problem of non-linear, periodic vibration of frames structures with clearances at supports is considered. The steady state responses of structures excited by harmonic forces are of particular interest. The periodic solutions are described using truncated Fourier series in time. The Fourier coefficients are determined from the non-linear amplitude equations. The matrix amplitude equation is derived with the help of the harmonic balance method. The unilateral constrains are taken into account in a course of determination of amplitude equation coefficients. The amplitude equation is treated as the equation with parameter and the frequency of excitation is chosen as the main parameter. The incremental-iterative procedure is used to solve the amplitude equation and to determine the response curves. Results of example calculations are also presented and briefly discussed.*

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1 Introduction

In this paper we consider the problem of non-linear, steady state vibration of frames and beams with clearances at supports. In particular, we analyse the structures excited by harmonic forces. These types of structures have a piecewise linear characteristic. As it is shown in [1], the behaviour of such structures could be strongly non-linear.

The static problems of the so-called slackened frame structures (i.e. structures with gaps at structural joints and supports) have been analysed by Gawęcki *et al.* (see [1,2]) where an advanced computational model is introduced.

The particular problem considered in this paper belongs to a wider class of dynamics of systems with non-smooth characteristics. The example of complex system with non-smooth characteristic is a multibody system that contains the colliding bodies. Recently, in paper [3] the advanced formulation of dynamic problems of such systems is presented. In this case, the motion equations are usually solved using the time integration methods.

The dynamic problems of structural systems with piecewise linear stiffness, such as beams or frames, are rarely considered. Only in [4,5], the steady state behaviour of beams are considered. Moreover, the multi-degree-of-freedom systems are analysed in papers [6,7] where the periodic responses of systems are considered.

Most of the recent research concerning impact oscillators is based on single-degree-of-freedom models. The one-degree-of-freedom systems with piecewise linear stiffness are extensively studied in many papers. An overview can be found in Bishop [8]. It was found that the dynamic behaviour of one-degree of freedom systems could be very complicated. In previous papers [9-11], the periodic, quasi-periodic and chaotic motions are analysed both theoretically and experimentally.

The paper is organised as follows. The assumptions and equations of motion are discussed in Section 2. In Section 3 the matrix amplitude equations are derived with the use of the harmonic balance method. The method of determination of response curves is described in Section 4. The results of example calculations are presented and discussed in Section 5. Finally, in Section 6 some conclusions are drawn.

2 Assumptions and equations of motion

Consider the elastic bar structure excited by harmonic forces. The gaps exist between the structure and some structure supports. However, it is assumed that the ordinary supports (i.e. supports, which can be treated as the two-side constraints,) secure the geometric immovability of considered structures. The frame or beam with all gaps open will be called the basic structure. Two kinds of gaps, the linear and angular ones, are taken into account in particular. The friction forces at supports are neglected. Moreover, the structure is massless and mass of the structure is lumped at some points. It is also assumed that there are no masses at points where the gaps can occur. Displacements of structures and gaps are small enough so the linear theory of kinematics can be used. An example of the considered systems is shown in Fig.1.

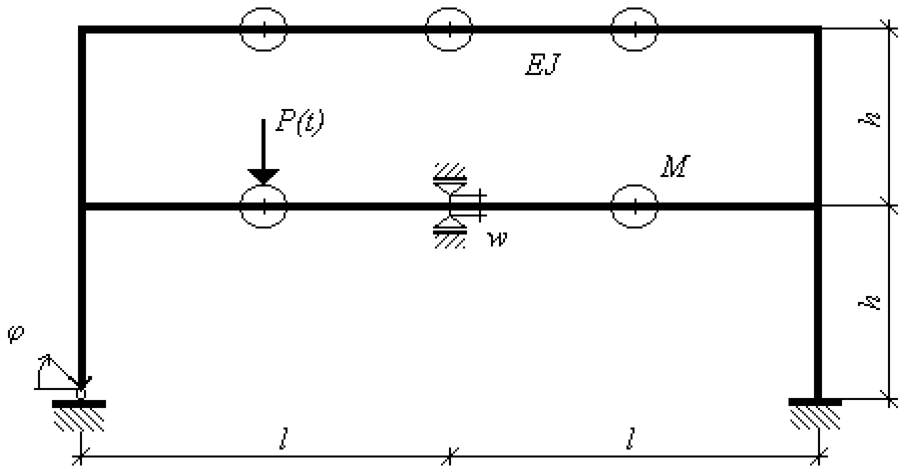

Fig.1 The example of frame with gaps (the angular φ and the linear *w)*

The unilateral conditions can be written in the following form [1]:

$$\mathbf{g}(t) = \mathbf{N}^T \mathbf{w}(t) - \mathbf{w}_o \leq \mathbf{0} \ , \tag{1}$$

$$sign\mathbf{r}(t) = sign\mathbf{w}(t) \ , \tag{2}$$

$$\mathbf{r}^T(t)\mathbf{g}(t) = 0 \ , \tag{3}$$

where $\mathbf{r}(t)$, $\mathbf{w}(t)$, $\mathbf{w}_o$, $\mathbf{N}$ are, respectively, the vector of support reactions, the vector of structure displacements at supports, the vector of limits gaps and the matrix of compatibility. The inequality (1) can be also written in the form:

$$\mathbf{w}_o^- \leq \mathbf{w}(t) \leq \mathbf{w}_o^+, \tag{1a}$$

where $\mathbf{w}_o^-$, $\mathbf{w}_o^+$ are the vectors of lower and upper limits of gaps, respectively. The finite element method is used to model the structure in a usual way.

Taking into account the above assumptions, we can write the equilibrium equation at time *t* in the form:

$$\hat{\mathbf{M}}\ddot{\hat{\mathbf{v}}}(t) + \hat{\mathbf{C}}\dot{\hat{\mathbf{v}}}(t) + \hat{\mathbf{K}}\hat{\mathbf{v}}(t) - \hat{\mathbf{p}}(t) - \hat{\mathbf{r}}(t) = \mathbf{0} \ , \tag{4}$$

where symbols $\hat{\mathbf{M}}$, $\hat{\mathbf{C}}$, $\hat{\mathbf{K}}$, $\hat{\mathbf{v}}(t)$, $\hat{\mathbf{p}}(t)$, $\hat{\mathbf{r}}(t)$ denote the mass and damping matrices, the stiffness matrix of the basic structure, the vector of nodal displacements of structure, the vector of nodal excitation forces and the vector of reactions at supports with gaps, respectively. Dots indicate differentiation with respect to time. In general, at time t, the vector $\tilde{\mathbf{v}}(t) = col(\mathbf{v}_d(t), \mathbf{v}_l(t), \mathbf{v}_a(t))$ contains three types of displacements, i.e. the displacements $\mathbf{v}_d(t)$ which cannot be in contact with support, the displacements $\mathbf{v}_l(t)$ which can be in contact but they are not at the present time, and the displacements $\mathbf{v}_a(t)$ which are currently in contact with supports. It is obvious that dimensions of vectors $\mathbf{v}_l(t)$ and $\mathbf{v}_a(t)$ change in time but in all cases $\mathbf{w}(t) = col(\mathbf{v}_l(t), \mathbf{v}_a(t))$. Now, taking into account this partition, the above equation could be also written in the following form:

$$\mathbf{M}\ddot{\mathbf{v}}_d(t) + \mathbf{C}\dot{\mathbf{v}}_d(t) + \mathbf{K}_{dd}\mathbf{v}_d(t) + \mathbf{K}_{dl}\mathbf{v}_l(t) + \mathbf{K}_{da}\mathbf{v}_a(t) - \mathbf{p}(t) = \mathbf{0} \ ,$$

$$\mathbf{K}_{ld}\mathbf{v}_d(t) + \mathbf{K}_{ll}\mathbf{v}_l(t) + \mathbf{K}_{la}\mathbf{v}_a(t) = \mathbf{0} \ ,$$

$$\mathbf{K}_{ad}\mathbf{v}_d(t) + \mathbf{K}_{al}\mathbf{v}_l(t) + \mathbf{K}_{aa}\mathbf{v}_a(t) - \mathbf{r}_a(t) = \mathbf{0} \qquad (5)$$

For the given vectors $\mathbf{v}_d(t)$, $\mathbf{v}_l(t)$ and $\mathbf{v}_a(t) = \mathbf{v}_{ao}$, where $\mathbf{v}_{ao}$ is the vector of currently closed gaps, the vector of reactions $\hat{\mathbf{r}}(t) = col(\mathbf{0},\ \mathbf{0},\ \mathbf{r}_a(t))$ could be determined from Equation (5.3). Using Equation (5.2), we obtain

$$\mathbf{v}_l(t) = -\mathbf{K}_{ll}^{-1}\mathbf{K}_{ld}\mathbf{v}_d(t) - \mathbf{K}_{ll}^{-1}\mathbf{K}_{la}\mathbf{v}_{ao} \ , \qquad (6)$$

and Equation (5.1) can be rewritten in the following form:

$$\mathbf{z}(t) \equiv \mathbf{M}\ddot{\mathbf{v}}_d(t) + \mathbf{C}\dot{\mathbf{v}}_d(t) + \mathbf{K}(t)\mathbf{v}_d(t) - \mathbf{p}(t) + \mathbf{f}_o(t) = \mathbf{0} \ , \qquad (7)$$

where

$$\mathbf{K}(t) = \mathbf{K}_{dd} - \mathbf{K}_{dl}\mathbf{K}_{ll}^{-1}\mathbf{K}_{ld} \ , \qquad \mathbf{f}_o(t) = \left(\mathbf{K}_{da} - \mathbf{K}_{dl}\mathbf{K}_{ll}^{-1}\mathbf{K}_{la}\right)\mathbf{v}_{ao} \ , \qquad (8)$$

and the residual vector $\mathbf{z}(t)$ vanishes in an equilibrium state. The motion equation (7) describes oscillations of structures in terms of nodal displacements which cannot be in contact with supports. The elements of the stiffness matrix $\mathbf{K}(t)$ are functions of time because dimensions of matrices $\mathbf{K}_{dl}$, $\mathbf{K}_{ll}$ and $\mathbf{K}_{ld}$ change in time. The non-linearity also comes into our problem through the vector $\mathbf{f}_o(t)$.

In this formulation the damping matrix $\mathbf{C}$ could be non-proportional but, in this paper, we assume that

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \ , \qquad (9)$$

where α, β are the proportionality factors and $\mathbf{K}$ is the stiffness matrix of the basic structure. This matrix can be determined using Equation (8.1), assuming earlier that all gaps are open, i.e. the dimension of vector $\mathbf{v}_a(t)$ is equal to zero.

3 Steady state solutions and amplitude equations

It is assumed that the vector of excitation forces can be described by:

$$\mathbf{p}(t) = \mathbf{p}_i^c \cos z_i \lambda t + \mathbf{p}_i^s \sin z_i \lambda t \ , \tag{10}$$

where λ and $T = 2\pi / \lambda$ are the fundamental frequency and the fundamental period of excitation, respectively, and $\mathbf{p}_i^c$, $\mathbf{p}_i^s$ are vectors of amplitudes of excitation forces. Moreover, z_i denotes the chosen integer number and *i=1,2,...,r*. The summation convention holds for repeated indices.

There are two different approaches for computation of the periodic response of a highly non-linear system. The first one is based on determining the monodromy matrix using the shooting method (see, [12]). The second approach, which we have followed in this paper, consists in a Fourier series expansion analysis. In this context, the harmonic balance method is widely use to solve non-linear problems under periodic excitation. The method is well-known from literature. The advanced computational formulation of the harmonic balance method is given in [13-15].

Following the harmonic balance method, the steady state vibration of considered structures is assumed to be described by the multi-harmonic function, i.e.

$$\mathbf{v}_d(t) = \mathbf{a}_i^c \cos z_l \lambda t + \mathbf{a}_i^s \sin z_l \lambda t \ , \tag{11}$$

where $\mathbf{a}_i^c$ and $\mathbf{a}_i^s$ are unknown vectors of amplitudes of vibration. Notice that the above form of solution is valid only for displacements which cannot be in contact with supports.

Sometimes, in non-linear systems, higher harmonics have a significant part in steady state responses of systems. For example, the complex behaviour of a system with piecewise-non-linear stiffness is reported in [10]. The proposed form of solution of motion equations takes into account these possibilities. However, it will be shown later that the response of a particular beam with a gap at support is periodic and that the influence of higher harmonics is small.

The assumed solution of motion equations is not exact and after introducing Equation (11) into (7) we obtain a vector of residuals $\mathbf{z}(t)$. The equilibrium conditions can be fulfilled only in a weak form. The weak forms of equilibrium conditions are:

$$\frac{1}{2T}\int_0^T \mathbf{z}(t)\cos z_l \lambda t dt = \mathbf{0} \ , \qquad \frac{1}{2T}\int_0^T \mathbf{z}(t)\sin z_l \lambda t dt = \mathbf{0} \ , \qquad l=1,2,...,r, \tag{12}$$

from which we obtain a following set of non-linear amplitude equations:

$$\left(\mathbf{K}_{li}^{cc} - z_i^2\lambda^2\mathbf{M}_{li}\right)\mathbf{a}_i^c + \left(\mathbf{K}_{li}^{cs} + z_i\lambda\mathbf{C}_{li}\right)\mathbf{a}_i^s + \mathbf{f}_l^{co} = \mathbf{p}_l^c \ ,$$

$$\left(\mathbf{K}_{li}^{sc} - z_i\lambda\mathbf{C}_{li}\right)\mathbf{a}_i^c + \left(\mathbf{K}_{li}^{ss} - z_i^2\lambda^2\mathbf{M}_{li}\right)\mathbf{a}_i^s + \mathbf{f}_l^{so} = \mathbf{p}_l^s \ . \tag{13}$$

for *l=1,2,...,r.*

Using the orthogonality properties of trigonometric functions we can write:

$$\mathbf{M}_{li} = \mathbf{M}\delta_{li} \ , \qquad \mathbf{C}_{li} = \mathbf{C}\delta_{li} \ . \tag{14}$$

where $\delta_{li} = 0$ for $i \neq l$ and $\delta_{li} = 1$ for $i = l$. The matrices $\mathbf{K}_{li}^{cc}$, $\mathbf{K}_{li}^{sc}$, $\mathbf{K}_{li}^{cs}$, $\mathbf{K}_{li}^{ss}$ and vectors $\mathbf{f}_l^{co}$, $\mathbf{f}_l^{so}$ depend on unilateral conditions which are active during the period of oscillations. These matrices and vectors are defined as follows:

$$\mathbf{K}_{li}^{cc} = \frac{1}{2T}\int_0^T \mathbf{K}(t)\cos z_l\lambda t\cos z_i\lambda t dt \ , \qquad \mathbf{K}_{li}^{cs} = \frac{1}{2T}\int_0^T \mathbf{K}(t)\cos z_l\lambda t\sin z_i\lambda t dt \ ,$$

$$\mathbf{K}_{li}^{sc} = \frac{1}{2T}\int_0^T \mathbf{K}(t)\sin z_l\lambda t\cos z_i\lambda t dt \ , \qquad \mathbf{K}_{li}^{ss} = \frac{1}{2T}\int_0^T \mathbf{K}(t)\sin z_l\lambda t\sin z_i\lambda dt \ , \tag{15}$$

$$\mathbf{f}_l^{co} = \frac{1}{2T}\int_0^T \mathbf{f}_o(t)\cos z_l\lambda t dt \ , \qquad \mathbf{f}_l^{so} = \frac{1}{2T}\int_0^T \mathbf{f}_o(t)\sin z_l\lambda t dt \ . \tag{16}$$

From these definitions it is easy to conclude that

$$\mathbf{K}_{il}^{cc} = \mathbf{K}_{li}^{cc} , \qquad \mathbf{K}_{il}^{ss} = \mathbf{K}_{li}^{ss} \ , \qquad \mathbf{K}_{il}^{cs} = \mathbf{K}_{li}^{sc} \ . \tag{17}$$

Taking into account the fact that the elements of matrix $\mathbf{K}(t)$ and vector $\mathbf{f}_o(t)$ are constant in intervals of time where the number of closed gaps is constant; we can rewrite Equations (15) and (16) in the following form:

$$\mathbf{K}_{li}^{cc} = \frac{1}{2T}\sum_k \mathbf{K}_k \int_{t_{kd}}^{t_{kg}} \cos z_l \lambda t \cos z_i \lambda t dt \ , \qquad \mathbf{K}_{li}^{cs} = \frac{1}{2T}\sum_k \mathbf{K}_k \int_{t_{kd}}^{t_{kg}} \cos z_l \lambda t \sin z_i \lambda t dt \ ,$$

$$\mathbf{K}_{li}^{sc} = \frac{1}{2T}\sum_k \mathbf{K}_k \int_{t_{kd}}^{t_{kg}} \sin z_l \lambda t \cos z_i \lambda t dt \ , \qquad \mathbf{K}_{li}^{ss} = \frac{1}{2T}\sum_k \mathbf{K}_k \int_{t_{kd}}^{t_{kg}} \sin z_l \lambda t \sin z_i \lambda t dt \ , \tag{18}$$

$$\mathbf{f}_l^{co} = \frac{1}{2T}\sum_k \mathbf{f}_k^o \int_{t_{kd}}^{t_{kg}} \cos z_l \lambda t dt \ , \qquad \mathbf{f}_l^{so} = \frac{1}{2T}\sum_k \mathbf{f}_k^o \int_{t_{kd}}^{t_{kg}} \sin z_l \lambda t dt \ , \tag{19}$$

where $\mathbf{K}_k$ and $\mathbf{f}_k^o$ are the matrix $\mathbf{K}(t)$ and the vector $\mathbf{f}_o(t)$, respectively, determined in the interval “k”. The symbols t_{kd} and t_{kg} denote the two successive time instances at which at least one gap closes or opens. Summation in (18) and (19) is over all intervals. It is easy to check that, for $i \neq l$, the results of integration can be written as follow:

$$\int_{t_{kd}}^{t_{kg}} \cos z_i \lambda t \cos z_l \lambda t dt = \frac{\sin(z_i + z_l)\lambda t_{kg} - \sin(z_i + z_l)\lambda t_{kd}}{2(z_i + z_l)\lambda} + \frac{\sin(z_i - z_l)\lambda t_{kg} - \sin(z_i - z_l)\lambda t_{kd}}{2(z_i - z_l)\lambda}$$

$$\int_{t_{kd}}^{t_{kg}} \sin z_i \lambda t \cos z_l \lambda t dt = \frac{\cos(z_i + z_l)\lambda t_{kd} - \cos(z_i + z_l)\lambda t_{kg}}{2(z_i + z_l)\lambda} + \frac{\cos(z_i - z_l)\lambda t_{kd} - \cos(z_i - z_l)\lambda t_{kg}}{2(z_i - z_l)\lambda}$$

$$\int_{t_{kd}}^{t_{kg}} \sin z_i \lambda t \sin z_l \lambda t dt = \frac{\sin(z_i - z_l)\lambda t_{kg} - \sin(z_i - z_l)\lambda t_{kd}}{2(z_i - z_l)\lambda} - \frac{\sin(z_i + z_l)\lambda t_{kg} - \sin(z_i + z_l)\lambda t_{kd}}{2(z_i + z_l)\lambda} \tag{20}$$

Moreover, for $i = l$ we obtain

$$\int_{t_{kd}}^{t_{kg}} \cos^2 z_l \lambda t dt = \frac{t_{kg} - t_{kd}}{2} + \frac{1}{4 z_l \lambda}\left(\sin 2 z_l \lambda t_{kg} - \sin 2 z_l \lambda t_{kd}\right) \ ,$$

$$\int_{t_{kd}}^{t_{kg}} \sin^2 z_l \lambda t dt = \frac{t_{kg} - t_{kd}}{2} - \frac{1}{4 z_l \lambda}\left(\sin 2 z_l \lambda t_{kg} - \sin 2 z_l \lambda t_{kd}\right) \ ,$$

$$\int_{t_{kd}}^{t_{kg}} \sin z_l \lambda t \cos z_l \lambda t dt = \frac{1}{2 z_l \lambda}\left(\sin^2 z_l \lambda t_{kg} - \sin^2 z_l \lambda t_{kd}\right) ,$$

$$\int_{t_{kd}}^{t_{kg}} \cos z_l \lambda t dt = \frac{1}{z_l \lambda}\left(\sin z_l \lambda t_{kg} - \sin z_l \lambda t_{kd}\right) ,$$

$$\int_{t_{kd}}^{t_{kg}} \sin z_l \lambda t dt = \frac{1}{z_l \lambda}\left(-\cos z_l \lambda t_{kg} + \cos z_l \lambda t_{kd}\right) . \tag{21}$$

The limits of intervals t_{kd} and t_{kg}, appearing in the above integrals, must be determined numerically. The following procedure is used. For a given time t, we can calculate the vector $\mathbf{v}_d(t)$, using Equation (11). The dimensions of vectors $\mathbf{v}_l(t)$ and $\mathbf{v}_{ao}$ must be determined in an iterative way. Having $\mathbf{v}_d(t)$, the vector $\mathbf{w}(t)$ can be calculated from

$$\mathbf{w}(t) = -\mathbf{K}_{ll}^{-1}\mathbf{K}_{ld}\mathbf{v}_d(t) . \tag{22}$$

Equation (22) follows from (6) if we assume that all gaps are open, which means that $\mathbf{w}(t) \equiv \mathbf{v}_l(t)$ and dimension of the vector $\mathbf{v}_a(t) = \mathbf{v}_{ao}$ is equal to zero.

The second possible choice is to take the vector $\mathbf{v}_a(t)$ from the previous time and calculate the vector $\mathbf{v}_l(t)$ from Equation (6). Now, we can verify the unilateral conditions (1) and we can find out which gaps are closed. The nodal displacements corresponding to closed gaps are equal to lower or upper limits of gaps and create the next approximation of vector $\mathbf{v}_a(t) = \mathbf{v}_{ao}$. This ends the first iteration. In the second iteration, having vectors $\mathbf{v}_d(t)$, $\mathbf{v}_{ao}$ and their respective dimensions, we can build new matrices $\mathbf{K}_{ll}$, $\mathbf{K}_{ld}$, $\mathbf{K}_{la}$ and calculate the new approximation of vector $\mathbf{v}_l(t)$, using Equation (6). If all elements of $\mathbf{v}_l(t)$ fulfil the unilateral conditions (1) the iteration process is completed. If not, the next iteration is performed. In our calculations, only two iterations are needed to obtain the correct vector $\mathbf{w}(t) = col(\mathbf{v}_l(t), \mathbf{v}_a(t))$ fulfilling the unilateral conditions.

Notice, the problem of finding the vectors $\mathbf{v}_l(t)$ and $\mathbf{v}_a(t)$ that, for given vector $\mathbf{v}_d(t)$, fulfil the unilateral conditions (1) – (3) and the equilibrium conditions (5.2) and (5.3) can be formulated and solved using the mathematical approach

The above procedure is repeated for a number of time instances closely spaced, in a range of integration *(0,T)*, on a time axis. If for two successive time instances t_n and t_{n+1} different gaps are closed, there exist a time instance t_i in which at least one gap closes or opens. This particular time can be determined using the method of bisection or the interpolation procedure described below. The vector of velocities $\dot{\mathbf{v}}_d(t)$ at times t_n and t_{n+1} can be calculated from

$$\dot{\mathbf{v}}_d(t) = -z_i \lambda \mathbf{a}_i^c \cos\lambda t + z_i \lambda \mathbf{a}_i^s \sin\lambda t \ . \tag{23}$$

Taking into account that $\dot{\mathbf{v}}_a(t) = \dot{\mathbf{v}}_{ao} = \mathbf{0}$ and after differentiation of Equation (5.2) with respect to time we obtain

$$\dot{\mathbf{v}}_l(t) = -\mathbf{K}_{ll}^{-1} \mathbf{K}_{ld} \dot{\mathbf{v}}_d(t) \ , \tag{24}$$

Now, it is assumed that for a sufficiently small interval of time $\Delta t = t_{n+1} - t_n$ the displacements $\mathbf{v}_l(t)$ can be approximated using the linear functions. The time t_i can be calculated using the following formulas:

$$t_i = t_n + \frac{v_{ao} - v_n}{\dot{v}_n} \ , \tag{25}$$

if the gap closes while

$$t_i = t_n + \Delta t + \frac{v_{ao} - v_{n+1}}{\dot{v}_{n+1}} \ , \tag{26}$$

when the gap opens. In Equations (25) and (26) v_n, v_{n+1}, $\dot{v}_n$, $\dot{v}_{n+1}$ are the displacements and velocities of node which can be in contact with the support and evaluated at time t_n and t_{n+1}, respectively. Moreover, v_{ao} denotes the limit of gap at this support. An idea of the proposed interpolation procedure is illustrated in Fig.2. The time t_i must be precisely determined because it has significant influence on calculation results. In this way, the unilateral conditions are taken into account during the process of calculation of matrices $\mathbf{K}_{li}^{cc}$, $\mathbf{K}_{li}^{cs}$, $\mathbf{K}_{li}^{sc}$, $\mathbf{K}_{li}^{ss}$ and vectors $\mathbf{f}_l^c$, $\mathbf{f}_l^s$.

The second possible approach to calculate the above mentioned quantities is simply to use the trapezoidal rule to calculate the integrals appearing in Equations (15) and (16). The appropriate formulas are:

$$\mathbf{K}_{li}^{cc} = \frac{1}{4T} \sum_{k=1}^{K} \Delta t_k \left(\mathbf{K}_{k-1} \cos z_l \lambda t_{k-1} \cos z_i \lambda t_{k-1} + \mathbf{K}_k \cos z_l \lambda t_k \cos z_i \lambda t_k \right) \ ,$$

$$\mathbf{K}_{li}^{cs} = \frac{1}{4T} \sum_{k=1}^{K} \Delta t_k \left(\mathbf{K}_{k-1} \cos z_l \lambda t_{k-1} \sin z_i \lambda t_{k-1} + \mathbf{K}_k \cos z_l \lambda t_k \sin z_i \lambda t_k \right) \ ,$$

$$\mathbf{K}_{li}^{ss} = \frac{1}{4T}\sum_{k=1}^{K} \Delta t_k \left(\mathbf{K}_{k-1} \sin z_l \lambda t_{k-1} \sin z_i \lambda t_{k-1} + \mathbf{K}_k \sin z_l \lambda t_k \sin z_i \lambda t_k \right) , \qquad (27)$$

$$\mathbf{f}_l^{co} = \frac{1}{4T}\sum_{k=1}^{K} \Delta t_k \left(\mathbf{f}_{k-1}^{o} \cos z_l \lambda t_{k-1} + \mathbf{f}_k^{o} \cos z_l \lambda t_k \right) ,$$

$$\mathbf{f}_l^{so} = \frac{1}{4T}\sum_{k=1}^{K} \Delta t_k \left(\mathbf{f}_{k-1}^{o} \sin z_l \lambda t_{k-1} + \mathbf{f}_k^{o} \sin z_l \lambda t_k \right) , \qquad (28)$$

where $\mathbf{K}_k = \mathbf{K}(t_k)$, $\mathbf{f}_k^o = \mathbf{f}^o(t_k)$, $\Delta t_k = t_k - t_{k-1}$. Approximately, one hundred intervals are sufficient to obtain correct results. In this case, the time instances where the stiffness of the system changes must also be precisely determined.

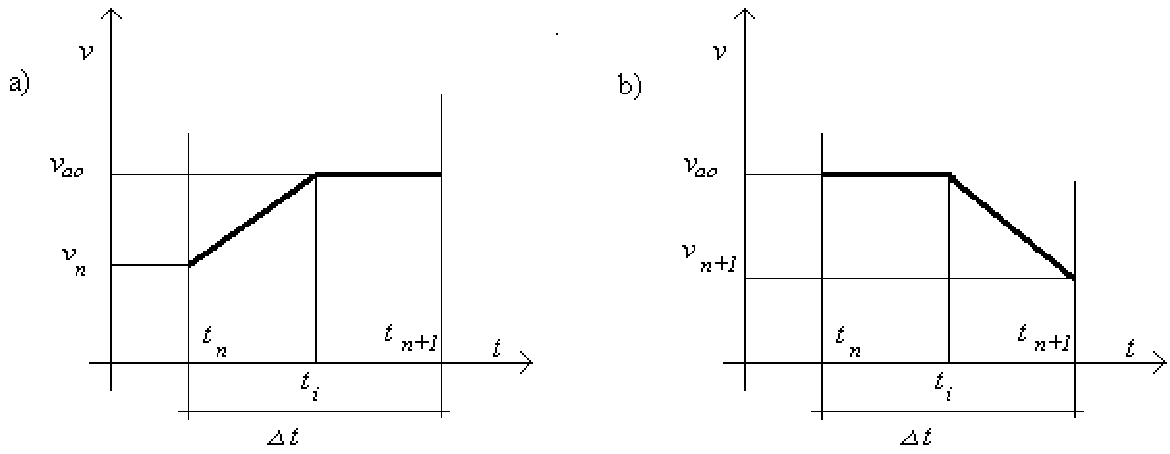

Fig.2 The illustration of approximation procedure

For the given frequency of excitation λ, the amplitude equations (13) are solved with respect to $\mathbf{a}_i^c$ and $\mathbf{a}_i^s$. The problem is non-linear because of the unilateral conditions (1) - (3). For this reason, the iterative procedure must be used. This procedure is described in the next section.

4 Determination of response curves

In many cases, the response curves must be determined in order to show the dynamic properties of the system under consideration. The response curve is obtained if the amplitude equations are solved for a set of values of excitation frequency, taken from a prescribed range (λ_a, λ_b). The incremental – iterative method is used to determine the response curves efficiently.

To be more consistent we introduce the following notation:

$$\tilde{\mathbf{a}} = col\left(\mathbf{a}_1^c, \mathbf{a}_1^s, \mathbf{a}_2^c, \mathbf{a}_2^s, \ldots\ldots, \mathbf{a}_r^c, \ \mathbf{a}_r^s\right) , \qquad \tilde{\mathbf{p}} = col\left(\mathbf{p}_1^c, \mathbf{p}_1^s, \mathbf{p}_2^c, \mathbf{p}_2^s, \ldots\ldots, \mathbf{p}_r^c, \mathbf{p}_r^s\right) ,$$

$$\tilde{\mathbf{f}}^o = col\left(\mathbf{f}_1^{co}, \mathbf{f}_1^{so}, \mathbf{f}_2^{co}, \mathbf{f}_2^{so}, \ldots\ldots, \mathbf{f}_r^{co}, \mathbf{f}_r^{so}\right) ,$$

$$\tilde{\mathbf{K}}_{li} = \begin{bmatrix} \mathbf{K}_{li}^{cc} & \mathbf{K}_{li}^{cs} \\ \mathbf{K}_{li}^{sc} & \mathbf{K}_{li}^{ss} \end{bmatrix} , \qquad \tilde{\mathbf{C}}_{ll} = z_l \begin{bmatrix} \mathbf{0} & \mathbf{C} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} , \qquad \tilde{\mathbf{M}}_{ll} = z_l^2 \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} ,$$

$$\tilde{\mathbf{M}} = diag\left(\tilde{\mathbf{M}}_{11}, \ \tilde{\mathbf{M}}_{22}, \ldots\ldots, \mathbf{M}_{rr}\right) , \qquad \tilde{\mathbf{C}} = diag\left(\tilde{\mathbf{C}}_{11}, \ \tilde{\mathbf{C}}_{22}, \ldots\ldots, \tilde{\mathbf{C}}_{rr}\right) ,$$

$$\tilde{\mathbf{K}} = \begin{bmatrix} \tilde{\mathbf{K}}_{11} & \tilde{\mathbf{K}}_{12} & \cdots\cdots & \tilde{\mathbf{K}}_{1l} & \cdots\cdots & \tilde{\mathbf{K}}_{1r} \\ \tilde{\mathbf{K}}_{21} & \tilde{\mathbf{K}}_{22} & \cdots\cdots & \tilde{\mathbf{K}}_{2l} & \cdots\cdots & \tilde{\mathbf{K}}_{2r} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tilde{\mathbf{K}}_{l1} & \tilde{\mathbf{K}}_{l2} & \cdots\cdots & \tilde{\mathbf{K}}_{ll} & \cdots\cdots & \tilde{\mathbf{K}}_{lr} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tilde{\mathbf{K}}_{r1} & \tilde{\mathbf{K}}_{r2} & \cdots\cdots & \tilde{\mathbf{K}}_{rl} & \cdots\cdots & \tilde{\mathbf{K}}_{rr} \end{bmatrix} . \tag{29}$$

Now, the amplitude equations (13) can be rewritten in the following compact form:

$$\left(\tilde{\mathbf{K}} - \lambda^2 \tilde{\mathbf{M}} + \lambda \tilde{\mathbf{C}}\right)\tilde{\mathbf{a}} + \tilde{\mathbf{f}}^o = \tilde{\mathbf{p}} \ . \tag{30}$$

The solutions of amplitude equations are represented by a sequence of excitation frequencies ${}^m\lambda$ and the amplitude vector ${}^m\tilde{\mathbf{a}}$ for *m=1,2,....*. For any incremental step, the vector ${}^m\tilde{\mathbf{a}}$ and ${}^m\lambda$ of the preceding step *m* are assumed to be given. At the beginning of the incremental procedure we can take λ_a far from the resonance regions. In non-resonance regions usually all unilateral conditions are non-active and the problem is linear.

For given ${}^m\lambda$ and ${}^m\tilde{\mathbf{a}}$, the frequency of excitation is increased by $\Delta\lambda$, i.e. ${}^{m+1}\lambda = {}^m\lambda + \Delta\lambda$ and the amplitude equations are solved in an iterative way, taking the vector ${}^m\tilde{\mathbf{a}}$ as the first approximation of the vector $\tilde{\mathbf{a}}$ in the iteration procedure. This completes the incremental step.

It is assumed that in a typical iteration we know an approximate solution denoted by $\tilde{\mathbf{a}}^i$. Now it is possible to calculate the matrix $\tilde{\mathbf{K}}$ and the vector $\tilde{\mathbf{f}}^o$, using the procedure described in the previous section. The next approximation of the solution of amplitude equations $\tilde{\mathbf{a}}^{i+1}$ is determined from Equation (30). The iterations are repeated until the following condition is fulfilled:

$$\left\| \tilde{\mathbf{a}}^{i+1} - \tilde{\mathbf{a}}^{i} \right\| \le \varepsilon_1 \left\| \tilde{\mathbf{a}}^{i+1} \right\| , \qquad (32)$$

where ε_1 is the assumed accuracy of calculations.

Using the above procedure, only the stable parts of the response curve can be determined. A more advanced procedure, similar to the continuation method described in [15], must be used if the whole response curve is needed. Research concerning this subject is in progress.

5 Results of example calculations

The response curve of the simply supported beam, shown in Fig.3, has been calculated and is presented in Fig.4. The data are as follows: the mass $M = 200.0$ kg, half of the beam length $a = 2.0$ m, the damping factor $c = 150.0$ Ns / m, the beam rigidity $EJ = 350550.0$ Nm^2 the amplitude of excitation force $P = 250.0$ N and the accuracy of calculation $\varepsilon_1 = 0.01$. The beam has a rotational clearance at left support. The upper and lower bounds of rotation at support are $\varphi_{10}^{+} = 0.005$ rad and $\varphi_{10}^{-} = -0.005$ rad, respectively.

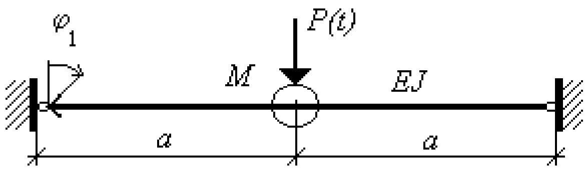

Fig.3 The simply supported beam with rotational gap

The results obtained by the method presented in this paper are shown as the thick line in Fig.4 and denoted as Curve 1. In [16], for this particular case we obtain the steady state solutions by means of the harmonic balance method and the continuation method. The three harmonics are taken into account in the Fourier series describing the steady state solution of motion equations and it was found that the influence of higher harmonics is very small. The results are shown as the thin curve denoted as Curve 2.

Moreover, the steady responses are obtained by means of the time integration method. The well-known Newmark method is used and the results are depicted in Fig. 4 by small crosses. It is obvious that there is strong agreement between results obtained with the use of all method. It is also demonstrated in [16] that it is possible to determine the whole response curve by the continuation method. The obtained results clearly indicate that the considered beam is a strongly non-linear system. In Fig. 4 two additional curves which represent the linear solutions, denoted as Curves 3 and 4, are shown for comparison. Curve 3 and 4 are the response curves of the simply supported beam and of the fixed-simply supported beam without gaps, respectively. The peak of non-linear response curve is lower then the peak of response curve for simply supported beam but higher than the response curve peak for fixed-simply supported beam. We also observe from Fig.4 that the non-linear response curve tends to be vertical in the upper part and that the main resonance region is between the natural frequencies of the simply supported beam and the fixed-simply supported beam, respectively.

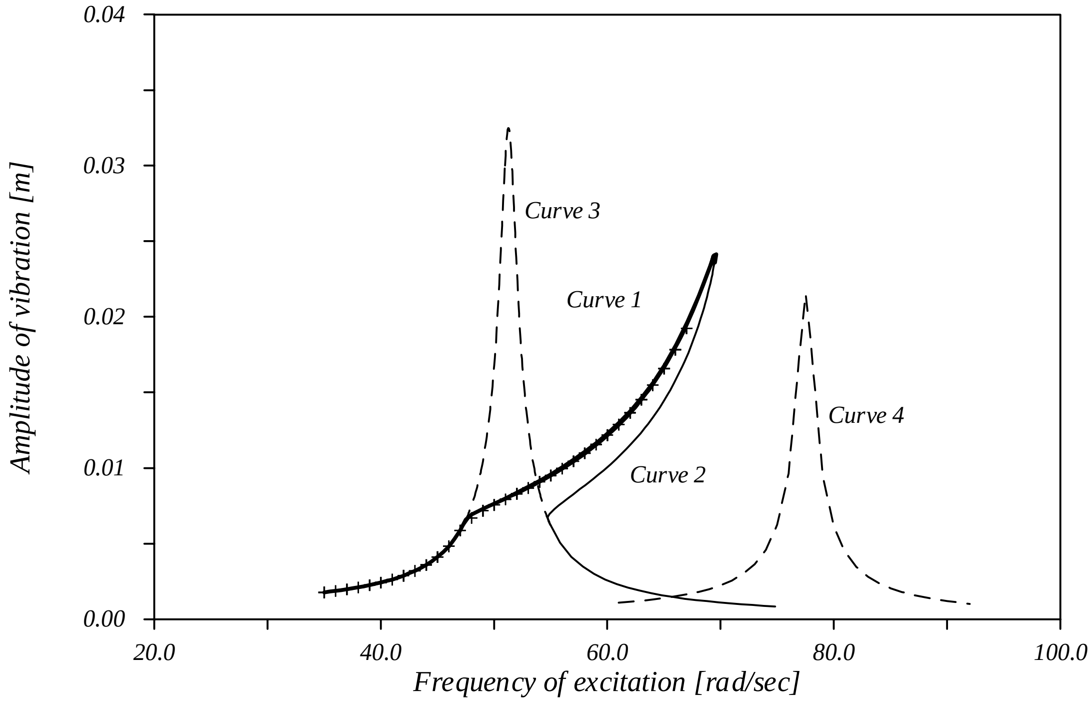

Fig.4 Response curve of simply supported beam with rotational gap at support

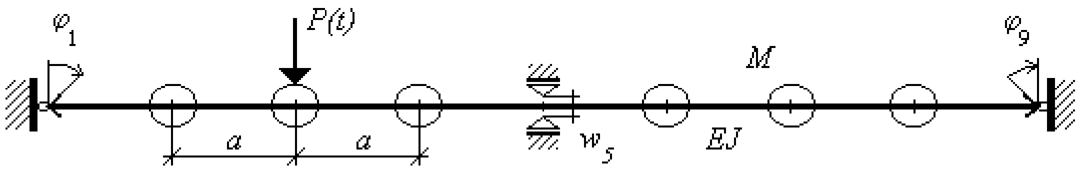

Fig.5 The beam with gaps at supports

The second example concerns the beam shown in Fig.5. The beam is divided into 8 equal finite elements of the length $a = 0.6$ m each; the granulated masses are also identical and $M = 10.0$ kg. The beam rigidity is $EJ = 2{,}000{,}000.0\ \mathrm{Nm}^2$, the non-dimensional modal damping factors of the first and second linear modes of vibration are $\gamma_1 = \gamma_2 = 0.008$. There are three gaps in this case, two rotational gaps at left and right support and $\varphi_{1o}^{+} = 0.04$ rad,

$\varphi_{1o}^{-} = -0.04$ rad, $\varphi_{9o}^{+} = 0.002$ rad, $\varphi_{9o}^{-} = -0.002$ rad, $w_{5o}^{+} = 0.02$ m and $w_{5o}^{-} = -0.02$ m, respectively. The excitation force acts on the second mass and $p_c = 100.0$ N while $p_s = 0.0$ N. Moreover, $\varepsilon_1 = 0.01$. Now the system with many degrees of freedom is analysed.

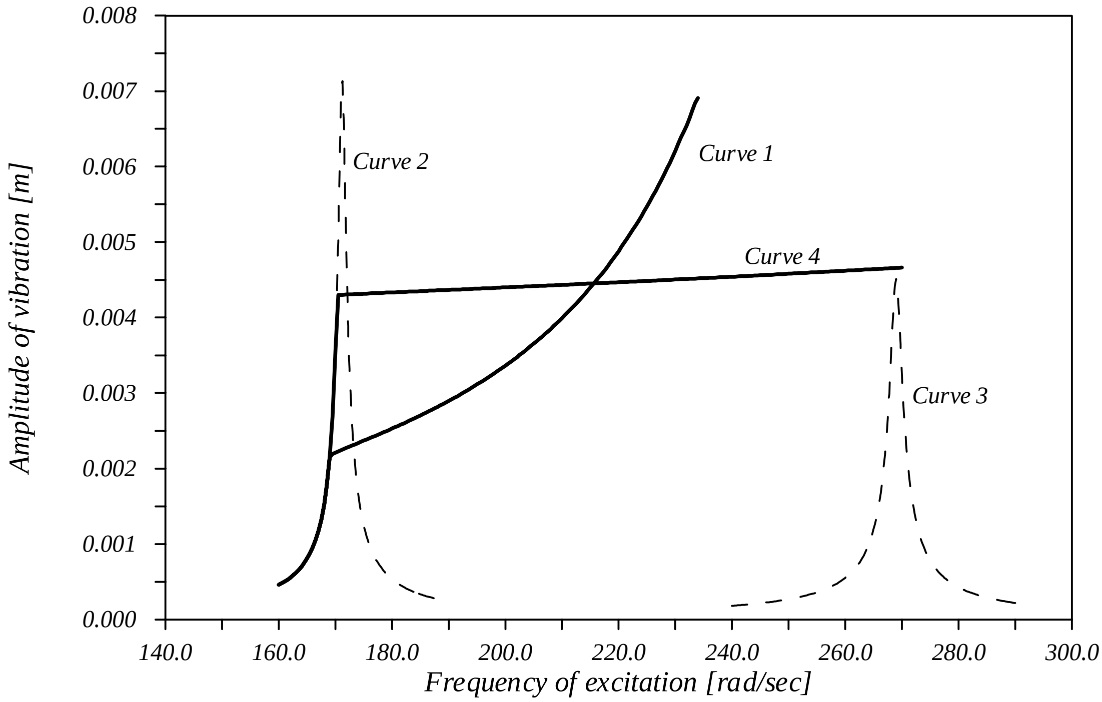

Fig.6 Example 2 - Response curves

The response curve is shown in Fig.6 and denoted as Curve 1. This picture shows the amplitude of vibration of the second mass versus the frequency of excitation. In this case, the gap at the right support is closed during a part of period of vibration. The curves denoted as Curves 2 and 3 are the response curves for the simply supported beam and the simply supported-fixed beam, respectively. The results are similar to those presented in Fig.4.

The dynamic behaviour of this beam changes significantly if the gap in the middle of the beam can be closed. The response curve for the considered beam with different gaps at supports is shown in Fig.6 and denoted as Curve 4. Now the limits of gaps are: $\varphi_{1o}^{+} = 0.04$ rad, $\varphi_{1o}^{-} = -0.04$ rad, $\varphi_{9o}^{+} = 0.04$ rad, $\varphi_{9o}^{-} = -0.04$ rad, $w_{5o}^{+} = 0.006$ m and $w_{5o}^{-} = -0.006$ m. We see that in this case the response curve is almost horizontal and does not grow as the previously discussed curves. The reason is that in this case the first natural frequency of a two-span, simply supported beam is far from the considered range of excitation frequency. The non-linear part of the response curve of the beam with gaps is similar to the response curve of the above mentioned two span beam in non-resonance region but with much greater amplitudes of vibration.

6 Concluding remarks

A method of analysis of steady state vibration of beams with gaps at supports is proposed in the paper. The harmonic balance method is used to derive the amplitude equations. The incremental-iterative procedure is used to determine the response curves. Results of example calculations are also presented and briefly discussed. These results show that the dynamic behaviour of the considered structures can be strongly non-linear and depend very much on reciprocal proportions of limit values of gaps. Only beams with symmetric limits of gaps are considered.

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References

[1] A. Gawęcki. *Elastic-plastic skeletal structures with clearances*. In: Ser. „Rozprawy', No 185, Poznan University of Technology Press, Poznan, (1987), (in Polish).

[2] A. Gawęcki, M.S. Kuczma and P. Kruger. Analysis of slackened skeletal structures by mathematical programming. *Computers and Structures*, **69**, 639-654, (1998).

[3] F. Pfeiffer. Unilateral problems of dynamics. *Archive of Applied Mechanics,* **69**, 503-527, (1999).

[4] E.L.B. van de Vorst, D.H. van Campen, A. De Kraker and R.H.B. Fey. Periodic solutions of a multi-dof beam system with impact. *Journal of Sound and Vibration,* **192**, 913-925, (1996).

[5] E.L.B. van de Vorst, M.F. Heertjes, D.H. van Campen, A. De Kraker and R.H.B. Fey. Experimental and numerical analysis of steady state behaviour of a beam system with impact. *Journal of Sound and Vibration,* **212**, 321-336, (1998).

[6] R.J. Comparin and R. Singh. Frequency response characteristics of a multi-degree-of-freedom system with clearances. *Journal of Sound and Vibration*, **142**, 101-124, (1990).

[7] D. Pun, S.L. Lau, S.S. Law and D.Q. Cao. Forced vibration analysis of a multidegree impact vibrator. *Journal of Sound and Vibration*, **213**, 447-466, (1998).

[8] S.R. Bishop. Impact oscillators. *Philosophical Transactions of Royal Society of London,* **347**, 345-448, (1994).

[9] S.L. Lau and W.S. Zhang. Nonlinear vibrations of piecewise-linear systems by incremental harmonic balance method. *Journal of Applied Mechanics*, **59**, 153-160, (1992).

[10] H.S. Choi and J.Y.K. Lou. Non-linear behavior and chaotic motion of an sdof system with piecewise-non-linear stiffness, *International Journal of Non-Linear Mechanics*, **26**, 461-473, (1991).

[11] M. Wiercigroch and V.W.T. Sin. Experimental study of a symmetrical piecewise base-excited oscillator. *Journal of Applied Mechanics,* **65**, 657-683, (1998).

[12] J.P. Meijaard. Direct determination of periodic solutions of the dynamical equations of flexible mechanisms and manipulators, *International Journal of Numerical Methods in Engineering,* **32**, 1691-1710, (1991).

[13] S.L. Lau and Y.K. Cheung. Amplitude incremental variational principle for non-linear vibration of elastic systems. *Journal of Applied Mechanics*, **48**, 959-964, (1981).

[14] A. Cardona, A. Lerusse and M. Geradin. Fast Fourier nonlinear vibration analysis. *Computational Mechanics,* **22**, 128-142, (1998).

[15] R. Lewandowski. Computational formulation for periodic vibration of geometrically nonlinear structures – part 1: theoretical background, *International Journal of Solids and Structures,* **34**, pp. 1925-1947, (1997).

[16] R. Lewandowski, J. Weiss. Analysis of steady state vibrations of beam with rotation clearence, In Proceedings of the XIV Polish Conference on computer methods in Mechanics, Rzeszów, Poland, 26-28 May 1999, pp. 221-222.