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The Computational Model for Non-Linear Vortex-Induced Vibration of Beams

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ABSTRACT: In this paper, the strips and finite element methods together with the Hartlen-Currie type vortex-shedding model are employed to study the vortex-induced vibration in a synchronisation region. The harmonic balance and continuation techniques are used to determine the periodic responses of beams. The amplitudes of vortex-induced vibration are determined by solving the amplitude equation with a parameter. The mean velocity is chosen as the main parameter. The continuation method is used to determine the response curve.

1. INTRODUCTION

Engineering structures exposed to wind flow often experience vortex-induced vibration. The effects of this vibration are the results of wind passing across a bluff body and forming an aerodynamic wake. As the frequency of shedding is approximately equal to one of natural frequencies of structure, the structure vibrates with large amplitudes. Vortex-induced vibration is found to be amplitude dependent, self limiting and highly sensitive to the structural damping level. In the synchronisation range, the vibration of structures are periodic but they are modulated ones outside this range. The recent, experimental results in this field are described by Brika and Laneville [1]. In civil engineering, we are mainly interested in the maximum vortex-induced response of structure. Therefore, the empirical or semi-empirical models that provide a reasonable approximation of the aeroelastic response are used. There are number of models of this kind. One model proposed by Hartlen and Currie [2] is found to be in good agreement with the experimental data.

The problem of vortex-induced vibration of beams was previously analysed by Baroush et al. [3] and Dul et al. [4] who used semi-empirical aerodynamic models and the time integration methods to determine the steady state vibration of beams. This approach requires extremely high computational effort because of small system damping. However, the computational cost can be drastically reduced by using the harmonic balance method as it is shown by Lewandowski [5] for the Scanlan model.

In this paper, the strips and finite element methods together with the Hartlen-Currie type vortex shedding model are used to formulate the computational model for studying the vortex-induced vibration of beams in a synchronisation region.

2. BRIEF DESCRIPTION OF HARTLEN-CURRIE MODEL

In paper [2] the elastically supported, rigid cylinder in air flow is considered. Cylinder motion is restricted to pure translation in the direction perpendicular to the flow direction and cylinder axis. The equations of motion for this model are derived by Hartlen and Currie in the following non-dimensional form:

$$\ddot{w} + 2\xi\dot{w} + w = a\omega_s^2 c_L \ , \tag{1}$$

$$\ddot{c}_L - \alpha\omega_s \dot{c}_L + \gamma\dot{c}_L^3 / \omega_s + \omega_s^2 c_L = b\dot{w} \, , \tag{2}$$

where $w,\ c_L, \xi,\ \omega_s,\ \alpha,\ \gamma,\ a,\ b$ are the cylinder displacement, the „hidden” aerodynamic variable interpreted as the lift coefficient, the damping factor for cylinder, the non-dimensional shedding frequency which is proportional to the wind mean velocity U, and some aerodynamic constants, respectively. The aerodynamic constants are determined experimentally. Equation 2 is the non-linear, van der Pole differential equation. More detailed description of the mentioned model is given in [2]. In next section, the Hartlen - Currie model will be extended for beams treated as systems with multi degrees of freedom.

3. EQUATIONS OF MOTION

The considered system (the beam and flow field) is divided into finite elements (beam) and strips (flow field). Each strip is parallel to the direction of undisturbed flow and has a width equal to the finite element length. The strips are also perpendicular to the finite elements. The main assumption is that flows in strips are mutually independent which means that the aerodynamic forces are induced only by flow in the associated strip.

The distribution of lift coefficient along the strip width is assumed in the following form:

$$c_L(x,t) = \mathbf{N}_L^T(x)\mathbf{c}_e(t) \ , \tag{3}$$

where $\mathbf{N}_L(x) = col(N_1(x), N_2(x))$, $N_1(x) = 1 - \eta$, $N_2(x) = \eta$, $\eta = x/l$, $\mathbf{c}_e = col(c_a, c_b)$ are the vector of shape functions and the vector of nodal parameters for strip, respectively. As we can see the continuity in the flow field is assumed.

The cross-wind transverse displacements w of the typical two node beam finite element with two degrees of freedom per node are described by using the Hermitan polynomials shape functions i.e.

$$w(x,t) = \mathbf{N}_b^T(x)\mathbf{w}_e(t) \ , \tag{4}$$

where $\mathbf{N}_b(x) = col(N_3(x), N_4(x), N_5(x), N_6(x))$ and $\mathbf{w}_e(t) = col(w_a, \varphi_a, w_b, \varphi_b)$ are the vector of beam shape functions and the vector of nodal parameters, respectively.

The kinetic and potential energy of finite element can be written in the form

$$T_b^e = \tfrac{1}{2}\dot{\mathbf{w}}_e^T \mathbf{M}_b^e \dot{\mathbf{w}}_e \ , \qquad W_b^e = \tfrac{1}{2}\mathbf{w}_e^T \mathbf{K}_b^e \mathbf{w}_e \ . \tag{5}$$

In above relations

$$\mathbf{M}_b^e = \int_0^l m(x)\mathbf{N}_b(x)\mathbf{N}_b^T(x)dx \ , \qquad \mathbf{K}_b^e = \int_0^l EJ(x)\mathbf{N}_b''(x)\mathbf{N}_b''^T(x)dx \ , \tag{6}$$

where $m(x)$, $EJ(x)$, l are the mass per unit length, the bending stiffness and the length of element, respectively.

The virtual work of non-conservative forces consists of damping term and the external excitation term. The damping term is represented by

$$\delta L_{bd}^e = \delta\mathbf{w}_e^T \mathbf{D}_b^e \dot{\mathbf{w}}_e \ , \tag{7}$$

where the damping matrix is given by $\mathbf{D}_b^e = \kappa_1\mathbf{M}_b^e + \kappa_2\mathbf{K}_b^e$ (κ_1, κ_2 are some parameters).

The aerodynamic external forces acting on beam are assumed to be similar in a mathematical form depicted in the Hartlen - Currie model so the virtual work of these forces can be written as

$$\int_{t_1}^{t_2} \boldsymbol{d}L_{ba}^{e} dt = \int_{t_1}^{t_2}\int_{0}^{l} \boldsymbol{d}w(x)\tfrac{1}{2}\,\boldsymbol{r}U^{2}(x)d(x)c_{L}(x,t)dxdt = \int_{t_1}^{t_2} \boldsymbol{w}_{s}^{2}\boldsymbol{d}\mathbf{w}_{e}^{T}\mathbf{S}_{L}^{e}\mathbf{c}_{e}(t)dt \ , \qquad (8)$$

where symbols $r,\ U(x),\ d(x),$ denote the air density, the mean velocity of wind and the cross-section characteristic dimension. The matrix $\mathbf{S}_{L}^{e}$ is defined by

$$\mathbf{S}_{L}^{e} = \frac{D^{3}p_{e}^{2}d_{e}}{8\boldsymbol{p}^{2}S^{2}}\int_{0}^{l}\mathbf{N}_{b}(x)\mathbf{N}_{L}^{T}(x)dx \ , \qquad (9)$$

where $\boldsymbol{w}_{s} = 2\boldsymbol{p}SU/D$, D is the reference cross-section dimension, S is the Strouhal number and d_{e} is the non-dimensional cross-section characteristic dimension.

Equation (2) in the Hartlen-Currie model describes the motion of some artificial variable which characterises the flow action in a globally way. This equation take into account only the primary characteristic deduced from experiments. However, many details connected with flow are omitted. From the mathematical point of view Eqn (2) could be understood as the motion equation of fictitious mechanical oscillator with non-linear damping characteristic. In order to make possible the weak formulation for whole considered system the ‘kinetic and potential energy’ and the ‘virtual work of non-conservative forces’ for the ficitous oscillators are also introduced. Of course, these quantities must be considered as some kind of functional which lead us to the counterparts of Eqn (2) in case of system with many degrees of freedom.

The ‘kinetic and potential energy’ for lift coefficient is defined as follows:

$$T_{L}^{e} = \tfrac{1}{2}\dot{\mathbf{c}}_{e}^{T}\mathbf{M}_{L}^{e}\dot{\mathbf{c}}_{e} \ , \qquad W_{L}^{e} = \tfrac{1}{2}\boldsymbol{w}_{s}^{2}\mathbf{c}_{e}^{T}\mathbf{K}_{L}^{e}\mathbf{c}_{e} \ , \qquad (10)$$

where

$$\mathbf{M}_{L}^{e} = \frac{1}{l}\int_{0}^{l}\mathbf{N}_{L}(x)\mathbf{N}_{L}^{T}(x)dx \ , \qquad \mathbf{K}_{L}^{e} = \frac{p_{e}^{2}}{ld_{e}^{2}}\int_{0}^{l}\mathbf{N}_{L}(x)\mathbf{N}_{L}^{T}(x)dx \ . \qquad (11)$$

The ‘virtual work of damping forces’ for ‘hidden variable’ is defined by:

$$\int_{t_1}^{t_2}\boldsymbol{d}L_{Ld}^{e}\,dt = \int_{t_1}^{t_2}\int_{0}^{l}\boldsymbol{d}c_{L}(x)(-\boldsymbol{a}\boldsymbol{w}_{s}\dot{c}_{L}(x,t) + \boldsymbol{g}\dot{c}_{L}^{3}(x,t)/\boldsymbol{w}_{s})dxdt =$$

$$\int_{t_1}^{t_2}\boldsymbol{d}\mathbf{c}_{e}^{T}\left[-\boldsymbol{w}_{s}\mathbf{D}_{L}^{e} + \boldsymbol{w}_{s}^{-1}\mathbf{D}_{NL}^{e}(\dot{\mathbf{c}}_{e},\dot{\mathbf{c}}_{e})\right]\dot{\mathbf{c}}_{e}(t)dt \ , \qquad (12)$$

where

$$\mathbf{D}_{L}^{e} = \frac{\boldsymbol{a}_{e}p_{e}}{ld_{e}}\int_{0}^{l}\mathbf{N}_{L}(x)\mathbf{N}_{L}^{T}(x)dx, \qquad (13)$$

$$\mathbf{D}_{NL}^{e}(\dot{\mathbf{c}}_{e},\dot{\mathbf{c}}_{e}) = \frac{\boldsymbol{g}_{e}d_{e}}{lp_{e}}\int_{0}^{l}\mathbf{N}_{L}^{T}(x)\dot{\mathbf{c}}_{e}(t)\dot{\mathbf{c}}_{e}^{T}(t)\mathbf{N}_{L}(x)\mathbf{N}_{L}(x)\mathbf{N}_{L}^{T}(x)dx \ .$$

(14)

The non-linear damping matrix $\mathbf{D}_{NL}^e(\dot{\mathbf{c}}_e, \dot{\mathbf{c}}_e)$ is the quadratic function of velocity of nodal parameters. In above relations the symbols α_e and γ_e denote the constants which must be determined experimentally. It is assumed here that they can be different for each strip.

The 'virtual work of external forces' is defined by

$$\int_{t_1}^{t_2} \delta L_{Lf}^e \, dt = \int_{t_1}^{t_2}\int_0^l \delta c_L(x) b \dot{w}(x,t) dx dt = \int_{t_1}^{t_2} \delta\mathbf{c}_e^T \mathbf{S}_b^e \dot{\mathbf{w}}_e(t) dt \ , \tag{15}$$

where

$$\mathbf{S}_b^e = \frac{b_e}{l}\int_0^l \mathbf{N}_L(x)\mathbf{N}_b^T(x) dx \ . \tag{16}$$

The equations of motion are derived on a basis of Hamilton principle, which states that

$$\int_{t_1}^{t_2} \left[\delta\left(T - W\right) + \delta L\right] dt = 0 \ , \tag{17}$$

where δ is the variational operator and T, W and δL denote the kinetic and potential energy of system and the virtual work of non-conservative forces, respectively. In our case

$$T = \sum_{e=1}^{n}\left(T_b^e + T_L^e\right) \ , \quad W = \sum_{e=1}^{n}\left(W_b^e + W_L^e\right) \ , \quad \delta L = \sum_{e=1}^{n}\left(\delta L_{bd}^e + \delta L_{ba}^e + \delta L_{Ld}^e + \delta L_{Lf}^e\right) \ . \tag{18}$$

Using the Hamilton principle we can derive the following equations of motion for the typical beam element and strip, respectively:

$$\mathbf{R}_b^e = \mathbf{M}_b^e \ddot{\mathbf{w}}_e(t) + \mathbf{D}_b^e \dot{\mathbf{w}}_e(t) + \mathbf{K}_b^e \mathbf{w}_e(t) - \omega_s^2 \mathbf{S}_L^e \mathbf{c}_e(t) \ , \tag{19}$$

$$\mathbf{R}_L^e = \mathbf{M}_L^e \ddot{\mathbf{c}}_e(t) - \omega_s \mathbf{D}_L^e \dot{\mathbf{c}}_e(t) + \omega_s^{-1} \mathbf{D}_{NL}^e(\dot{\mathbf{c}}_e(t), \dot{\mathbf{c}}_e(t))\dot{\mathbf{c}}_e(t) + \omega_s^2 \mathbf{K}_L^e \mathbf{c}_e(t) - \mathbf{S}_b^e \dot{\mathbf{w}}_e(t) \ . \tag{20}$$

After assembling procedure the motion equations for the entire system can be written as:

$$\mathbf{R}_b(t) = \mathbf{M}_b \ddot{\mathbf{w}}(t) + \mathbf{D}_b \dot{\mathbf{w}}(t) + \mathbf{K}_b \mathbf{w}(t) - \omega_s^2 \mathbf{S}_L \mathbf{c}(t) = \mathbf{0} \ , \tag{21}$$

$$\begin{aligned} \mathbf{R}_L(t) = \mathbf{M}_L \ddot{\mathbf{c}}(t) - \omega_s \mathbf{D}_L \dot{\mathbf{c}}(t) + \omega_s^{-1} \mathbf{D}_{NL}(\dot{\mathbf{c}}(t), \dot{\mathbf{c}}(t))\dot{\mathbf{c}}(t) + {} \\ \omega_s^2 \mathbf{K}_L \mathbf{c}(t) - \mathbf{S}_b \dot{\mathbf{w}}(t) = \mathbf{0} \ , \end{aligned} \tag{22}$$

where $\mathbf{M}_b$, $\mathbf{M}_L$, $\mathbf{D}_b$, $\mathbf{D}_L$, $\mathbf{D}_{NL}(\dot{\mathbf{c}}(t), \dot{\mathbf{c}}(t))$, $\mathbf{K}_b$, $\mathbf{K}_L$, $\mathbf{S}_b$, $\mathbf{S}_L$, $\mathbf{w}(t)$, $\mathbf{c}(t)$ are the global counterparts of previously defined, on a level of element and strip, matrices and vectors. The residual vectors $\mathbf{R}_b(t)$ and $\mathbf{R}_L(t)$ vanish in an equilibrium state.

4. AMPLITUDE EQUATIONS

The steady state, periodic response of system can be described in a first approximation by

$$\mathbf{w}(t) = \mathbf{w}_c \cos\omega t + \mathbf{w}_s \sin\omega t \ , \qquad \mathbf{w}_e(t) = \mathbf{w}_{ce} \cos\omega t + \mathbf{w}_{se} \sin\omega t \ , \tag{23}$$

$$\mathbf{c}(t) = \mathbf{c}_c \cos\omega t + \mathbf{c}_s \sin\omega t \ , \qquad \mathbf{c}_e(t) = \mathbf{c}_{ce} \cos\omega t + \mathbf{c}_{se} \sin\omega t \ , \tag{24}$$

where $\mathbf{w}_c$, $\mathbf{w}_s$, $\mathbf{c}_c$, $\mathbf{c}_s$, $\mathbf{w}_{ce}$, $\mathbf{w}_{se}$, $\mathbf{c}_{ce}$, $\mathbf{c}_{se}$ are the unknown vectors of harmonic amplitudes of nodal parameters of beam and strips on a level of entire system and the finite element and strip, respectively. Also the frequency of oscillation ω is the unknown quantity.

In this work, the solution with only one harmonic is taken into account because the results of experiments show that it is sufficiently accurate in the synchronisation region.

The in time Galerkin procedure is used to derive the amplitude equations. These equations follow from the Galerkin conditions which in our case state that

$$\frac{2}{T}\int_0^T \mathbf{R}_b(t)\cos wt\,dt = \mathbf{0}\ , \qquad \frac{2}{T}\int_0^T \mathbf{R}_b(t)\sin wt\,dt = \mathbf{0}\ , \tag{25}$$

$$\frac{2}{T}\int_0^T \mathbf{R}_L(t)\cos wt\,dt = \mathbf{0}\ , \qquad \frac{2}{T}\int_0^T \mathbf{R}_L(t)\sin wt\,dt = \mathbf{0}\ , \tag{26}$$

where $T = 2p/w$ denotes the period of limit cycle. The residual vectors $\mathbf{R}_b(t)$, $\mathbf{R}_L(t)$ appearing in Eqs (25) and (26) are determined by introducing the assumed solution of motion equations into Eqs (21) and (22).

After calculation of resulting integrals, from the Galerkin conditions one obtains the following set of non-linear algebraic equations:

$$(\mathbf{K}_b - w^2\mathbf{M}_b)\mathbf{w}_c + w\mathbf{D}_b\mathbf{w}_s - w_s^2\mathbf{S}_L\mathbf{c}_c = \mathbf{0}\ , \tag{27}$$

$$-w\mathbf{D}_b\mathbf{w}_c + (\mathbf{K}_b - w^2\mathbf{M}_b)\mathbf{w}_s - w_s^2\mathbf{S}_L\mathbf{c}_s = \mathbf{0}\ , \tag{28}$$

$$(w_s^2\mathbf{K}_L - w^2\mathbf{M}_L)\mathbf{c}_c - ww_s\mathbf{D}_L\mathbf{c}_s + \frac{3}{4}w^3w_s^{-1}\left[\mathbf{D}_{NL}(\mathbf{c}_c,\mathbf{c}_c) + \mathbf{D}_{NL}(\mathbf{c}_s,\mathbf{c}_s)\right]\mathbf{c}_s - w\mathbf{S}_b\mathbf{w}_s = \mathbf{0}\ , \tag{29}$$

$$ww_s\mathbf{D}_L\mathbf{c}_c - \frac{3}{4}w^3w_s^{-1}\left[\mathbf{D}_{NL}(\mathbf{c}_c,\mathbf{c}_c) + \mathbf{D}_{NL}(\mathbf{c}_s,\mathbf{c}_s)\right]\mathbf{c}_c + (w_s^2\mathbf{K}_L - w^2\mathbf{M}_L)\mathbf{c}_s + w\mathbf{S}_b\mathbf{w}_c = \mathbf{0}\ . \tag{30}$$

The non-linear matrix $\mathbf{D}_{NL}(\mathbf{c}_c,\mathbf{c}_s)$ is a quadratic function of strip nodal parameters and the elements of this matrix are defined on a strip level as follow:

$$\mathbf{D}_{NL}^{e}(\mathbf{c}_c,\mathbf{c}_s) = \frac{g_e d_e}{60p_e}\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\ , \tag{31}$$

where

$$d_{11} = 12c_{ca}c_{sa} + 3(c_{ca}c_{sb} + c_{cb}c_{sa}) + 2c_{cb}\mathrm{c}_{\mathrm{sb}}\ ,\quad d_{22} = 2c_{ca}c_{sa} + 3(c_{ca}c_{sb} + c_{cb}c_{sa}) + 12c_{cb}\mathrm{c}_{\mathrm{sb}}$$

$$d_{12} = d_{21} = 3c_{ca}c_{sa} + 2(c_{ca}c_{sb} + c_{cb}c_{sa}) + 3c_{cb}c_{sb}\ .$$

Using the following notation:

$$\tilde{\mathbf{M}}_b = \begin{bmatrix} \mathbf{M}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_b \end{bmatrix},\quad \tilde{\mathbf{K}}_b = \begin{bmatrix} \mathbf{K}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_b \end{bmatrix},\quad \tilde{\mathbf{D}}_b = \begin{bmatrix} \mathbf{0} & \mathbf{D}_b \\ -\mathbf{D}_b & \mathbf{0} \end{bmatrix},\quad \tilde{\mathbf{S}}_L = \begin{bmatrix} \mathbf{S}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_L \end{bmatrix}, \tag{32}$$

$$\tilde{\mathbf{M}}_L = \begin{bmatrix} \mathbf{M}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_L \end{bmatrix},\quad \tilde{\mathbf{K}}_L = \begin{bmatrix} \mathbf{K}_L & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_L \end{bmatrix},\quad \tilde{\mathbf{D}}_L = \begin{bmatrix} \mathbf{0} & \mathbf{D}_L \\ -\mathbf{D}_L & \mathbf{0} \end{bmatrix},\quad \tilde{\mathbf{S}}_b = \begin{bmatrix} \mathbf{0} & \mathbf{S}_b \\ -\mathbf{S}_b & \mathbf{0} \end{bmatrix}, \tag{33}$$

$$\tilde{\mathbf{D}}_{NL} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_{NL}(\mathbf{c}_c,\mathbf{c}_c) + \mathbf{D}_{NL}(\mathbf{c}_s,\mathbf{c}_s) \\ -\mathbf{D}_{NL}(\mathbf{c}_c,\mathbf{c}_c) - \mathbf{D}_{NL}(\mathbf{c}_s,\mathbf{c}_s) & \mathbf{0} \end{bmatrix}, \tag{34}$$

$$\tilde{\mathbf{w}} = col(\mathbf{w}_c,\mathbf{w}_s)\ ,\qquad \tilde{\mathbf{c}} = col(\mathbf{c}_c,\mathbf{c}_s)\ , \tag{35}$$

the mentioned amplitude equations can be rewritten in the compact form of first order:

$$\tilde{\mathbf{G}}_b(\tilde{\mathbf{w}},\tilde{\mathbf{c}},w,w_s) = (\tilde{\mathbf{K}}_b - w^2\tilde{\mathbf{M}}_b + w\tilde{\mathbf{D}}_b)\tilde{\mathbf{w}} - w_s^2\tilde{\mathbf{S}}_L\tilde{\mathbf{c}} = \mathbf{0} \quad , \tag{36}$$

$$\begin{aligned}&\tilde{\mathbf{G}}_L(\tilde{\mathbf{w}},\tilde{\mathbf{c}},w,w_s) = -w\tilde{\mathbf{S}}_b\tilde{\mathbf{w}} + \\ &\left[w_s^2\tilde{\mathbf{K}}_L - w^2\tilde{\mathbf{M}}_L - ww_s\tilde{\mathbf{D}}_L + \tfrac{3}{4}w^3w_s^{-1}\tilde{\mathbf{D}}_{NL}(\tilde{\mathbf{c}},\tilde{\mathbf{c}})\right]\tilde{\mathbf{c}} = \mathbf{0} \quad .\end{aligned} \tag{37}$$

The most compact form is given by

$$\tilde{\mathbf{G}}(w,w_s,\mathbf{a}) = \tilde{\mathbf{K}}(w,w_s,\mathbf{c})\tilde{\mathbf{a}} \quad , \tag{38}$$

where

$$\tilde{\mathbf{K}}(w,w_s,\tilde{\mathbf{c}}) = \begin{bmatrix} (\tilde{\mathbf{K}}_b - w^2\tilde{\mathbf{M}}_b + w\tilde{\mathbf{D}}_b) & -w_s^2\tilde{\mathbf{S}}_L \\ -w\tilde{\mathbf{S}}_b & (w_s^2\tilde{\mathbf{K}}_L - w^2\tilde{\mathbf{M}}_L - ww_s\tilde{\mathbf{D}}_L + \tfrac{3}{4}w^3w_s^{-1}\tilde{\mathbf{D}}_{NL}(\tilde{\mathbf{c}},\tilde{\mathbf{c}})) \end{bmatrix},$$

$$\tilde{\mathbf{G}} = col(\tilde{\mathbf{G}}_b,\tilde{\mathbf{G}}_L) \quad , \qquad \tilde{\mathbf{a}} = col(\tilde{\mathbf{w}},\tilde{\mathbf{c}}) \quad . \tag{39}$$

5. CONTINUATION PROCEDURE

Equation (38) is solved for different w_s (or *U* due to relation $w_s = 2pSU/D$) by using the continuation method. It means that the wind velocity is chosen as a main parameter. Obviously, from Eqn (38) $\tilde{\mathbf{a}} = \mathbf{0}$ is the solution of amplitude equation. Moreover, it is easy to verify that if w and $\tilde{\mathbf{a}} \neq \mathbf{0}$ are, for the particular *U*, the solution of amplitude equation then w, and $-\tilde{\mathbf{a}}$ are also the solution. It is follow from facts that the amplitude equation is homogenous and the non-linear matrices of type $\mathbf{D}_{NL}(\mathbf{c}_c,\mathbf{c}_s)$ are the quadratic functions of amplitudes of lift coefficients $\mathbf{c}_c$ and $\mathbf{c}_s$.

The trivial solution could be represented on the amplitude of vibration - wind velocity diagram or the amplitude of lift coefficient - wind velocity diagram as a line coinciding with w_s (or *U*) axis. On this line the bifurcation point exists. An additional branch representing the non-trivial solution of Eqn (38) emanates from the bifurcation point. This branch has been determined by means of the continuation method.

However, at the beginning the bifurcation point must be determined. It is not a trivial problem. In this case, the amplitude equations are reduced to

$$(\tilde{\mathbf{K}}_b - w^2\tilde{\mathbf{M}}_b + w\tilde{\mathbf{D}}_b)\tilde{\mathbf{w}} - w_s^2\tilde{\mathbf{S}}_L\tilde{\mathbf{c}} = \mathbf{0} \quad , \tag{40}$$

$$-w\tilde{\mathbf{S}}_b\tilde{\mathbf{w}} + \left[w_s^2\tilde{\mathbf{K}}_L - w^2\tilde{\mathbf{M}}_L - ww_s\tilde{\mathbf{D}}_L\right]\tilde{\mathbf{c}} = \mathbf{0} \quad . \tag{41}$$

In the above homogenous equations there are *n+1* unknowns (i.e. w, $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{c}}$) because w_s is treated as the parameter. However, since the considered dynamic system is autonomous, one of the Fourier coefficients can be fixed (e.g. $w_{c(s),i} = 0$ or $c_{c(s),i} = 0$), which causes only a shift of response on the time axis. After, introducing the notation

$$\tilde{\mathbf{K}}(w) = \begin{bmatrix} (\tilde{\mathbf{K}}_b - w^2\tilde{\mathbf{M}}_b + w\tilde{\mathbf{D}}_b) & -w_s^2\tilde{\mathbf{S}}_L \\ -w\tilde{\mathbf{S}}_b & (w_s^2\tilde{\mathbf{K}}_L - w^2\tilde{\mathbf{M}}_L - ww_s\tilde{\mathbf{D}}_L) \\ \multicolumn{2}{c}{0\;\;0\;\;0\ldots\ldots\ldots\ldots1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots0\;\;0} \end{bmatrix} \quad , \tag{42}$$

we can rewrite Eqs (40) and (41) in the form

$$\tilde{\mathbf{K}}(w)\tilde{\mathbf{a}} = \mathbf{0} \quad , \tag{43}$$

where the additional row contains the coefficients of the autonomous condition $\tilde{a}_j = 0$. The dimensions of matrix $\tilde{\mathbf{K}}(w)$ are *(n+1)xn*. Now, for a moment, the frequency of oscillation w is also treated as the parameter in Eqn (43). This matrix equation is the underdetermined system of homogenous algebraic equations. The theory of these equations tells us that the non-trivial solution exists if a rank of matrix $\tilde{\mathbf{K}}(w)$ is equal *n-1*. Moreover, in this case, there are infinitely many solutions but all of these are proportional to one reference solution. In general, the rank of matrix $\tilde{\mathbf{K}}(w)$ is *n*. Only for the particular values of $w_s = w_{s,cr}$ and $w = w_{cr}$, which define the bifurcation point on the response diagram, the rank of $\tilde{\mathbf{K}}(w)$ is equal *n-1*. These particular values $w_{s,cr}$ and w_{cr} are determined by using the Gauss - Jordan transformation. If the rank of considered matrix is equal *n-1* then as the result of application of this process to matrix $\tilde{\mathbf{K}}(w)$ one row with all elements equal zero is obtained. Simultaneously, after the Gauss - Jordan transformation each remaining row contains zero elements except one which is equal one. The appropriately scaled non-trivial solution of Eqn (43) is taken as the initial approximation of first point on the branch which emanates from the bifurcation point.

The non-trivial solution of amplitude equation is represented by a sequence of vortex-shedding frequency, frequency of periodic response and the amplitude vectors, i.e. ${}^m w_s,\ {}^m w,\ {}^m\tilde{\mathbf{a}}$ for *m=1,2,...* For any incremental step, the vector ${}^m\tilde{\mathbf{a}}$ and ${}^m w_s,\ {}^m w$ of the proceeding step is assumed to be given. The purpose of an incremental process is to find the following increments $\Delta w_s,\ \Delta w,\ \Delta\tilde{\mathbf{a}}$ which can be accumulated to yield

$${}^{m+1}w_s = {}^m w_s + \Delta w_s, \qquad {}^{m+1}w = w + \Delta w, \qquad {}^{m+1}\tilde{\mathbf{a}} = {}^m\tilde{\mathbf{a}} + \Delta\tilde{\mathbf{a}} \ . \tag{44}$$

The previously mentioned autonomous condition and the following constraint equation

$$\Delta\tilde{\mathbf{w}}^T\Delta\tilde{\mathbf{w}} / m_b^2 + \Delta\tilde{\mathbf{c}}^T\Delta\tilde{\mathbf{c}} / m_L^2 = (\Delta s)^2 \ , \tag{45}$$

are added to the matrix amplitude equation. In Eqn (45) the symbols $m_b,\ m_L$ mean the scaling factors. These set of non-linear equations can be solved only by an iterative procedure. Suppose, after iteration *i* we know an approximation of solution denoted by $w_s^i,\ w^i,\ \tilde{\mathbf{a}}^i$. The iterative change of all unknowns quantities $dw_s,\ dw,\ d\tilde{\mathbf{a}}$ are governed by the following equation

$$\tilde{\mathbf{K}}_t(w_s^i, w^i, \tilde{\mathbf{a}}^i)d\tilde{\mathbf{a}} = -\tilde{\mathbf{G}}(w_s^i, w^i, \tilde{\mathbf{a}}^i) - \tilde{\mathbf{H}}(w_s^i, w^i, \tilde{\mathbf{a}}^i)dw - \tilde{\mathbf{F}}(w_s^i, w^i, \tilde{\mathbf{a}}^i)dw_s \,, \tag{46}$$

where $\tilde{\mathbf{K}}_t(w_s^i, w^i, \tilde{\mathbf{a}}^i)$, $\tilde{\mathbf{H}}(w_s^i, w^i, \tilde{\mathbf{a}}^i)$ and $\tilde{\mathbf{F}}(w_s^i, w^i, \tilde{\mathbf{a}}^i)$ are the matrix and vectors of first derivatives of $\tilde{\mathbf{G}}$ with respect to $\tilde{\mathbf{a}}$, w and w_s, respectively. These matrices are given by:

$$\tilde{\mathbf{K}}_t = \begin{bmatrix} (\tilde{\mathbf{K}}_b - w^2\tilde{\mathbf{M}}_b + w\tilde{\mathbf{D}}_b) & -w_s^2\tilde{\mathbf{S}}_L \\ -w\tilde{\mathbf{S}}_b & (w_s^2\tilde{\mathbf{K}}_L - w^2\tilde{\mathbf{M}}_L - ww_s\tilde{\mathbf{D}}_L + \frac{3}{4}w^3 w_s^{-1}\tilde{\mathbf{Z}}_{NL}(\tilde{\mathbf{c}},\tilde{\mathbf{c}})) \end{bmatrix}, \tag{47}$$

$$\tilde{\mathbf{H}}(w_s^i, w^i, \tilde{\mathbf{a}}^i) = \begin{bmatrix} (-2w\tilde{\mathbf{M}}_b + \tilde{\mathbf{D}}_b)\tilde{\mathbf{w}} \\ (-2w\tilde{\mathbf{M}}_L - w_s\tilde{\mathbf{D}}_L + \frac{9}{4}w^2 w_s^{-1}\tilde{\mathbf{D}}_{NL}(\tilde{\mathbf{c}},\tilde{\mathbf{c}}))\tilde{\mathbf{c}} - \tilde{\mathbf{S}}_b\tilde{\mathbf{w}} \end{bmatrix}, \tag{48}$$

$$\tilde{\mathbf{F}}(w_s^i, w^i, \tilde{\mathbf{a}}^i) = \begin{bmatrix} -2w_s\tilde{\mathbf{S}}_L\tilde{\mathbf{c}} \\ (2w_s\tilde{\mathbf{K}}_L - w\tilde{\mathbf{D}}_L - \frac{3}{4}w^3 w_s^{-2}\tilde{\mathbf{D}}_{NL}(\tilde{\mathbf{c}},\tilde{\mathbf{c}}))\tilde{\mathbf{c}} \end{bmatrix}, \tag{49}$$

where

$$\tilde{\mathbf{Z}}_{NL}(\tilde{\mathbf{c}},\tilde{\mathbf{c}})=\begin{bmatrix} 2\mathbf{D}_{NL}(\mathbf{c}_c,\mathbf{c}_s) & \mathbf{D}_{NL}(\mathbf{c}_c,\mathbf{c}_c)+3\mathbf{D}_{NL}(\mathbf{c}_s,\mathbf{c}_s) \\ -\mathbf{D}_{NL}(\mathbf{c}_c,\mathbf{c}_c)-3\mathbf{D}_{NL}(\mathbf{c}_s,\mathbf{c}_s) & -2\mathbf{D}_{NL}(\mathbf{c}_c,\mathbf{c}_s) \end{bmatrix} . \tag{50}$$

The vector $d\tilde{\mathbf{a}}$ can be written as a sum of three components

$$d\tilde{\mathbf{a}} = d\tilde{\mathbf{a}}_1 + dw\,d\tilde{\mathbf{a}}_2 + dw_s d\tilde{\mathbf{a}}_3 \ , \tag{51}$$

where

$$d\tilde{\mathbf{a}}_1 = -\tilde{\mathbf{K}}_t^{-1}\tilde{\mathbf{G}}(w_s^i, w^i, \tilde{\mathbf{a}}^i) \ , \quad d\tilde{\mathbf{a}}_2 = -\tilde{\mathbf{K}}_t^{-1}\mathbf{H}(w_s^i, w^i, \tilde{\mathbf{a}}^i) \ , \quad d\tilde{\mathbf{a}}_3 = -\tilde{\mathbf{K}}_t^{-1}\mathbf{F}(w_s^i, w^i, \tilde{\mathbf{a}}^i) \ . \tag{52}$$

Substituting the total increment of $\tilde{\mathbf{a}}$ into the constrain equation (45) and the autonomous condition $\tilde{a}_j = 0$ we receive the following equations for dw and dw_s

$$h_1 dw + h_2 dw_s + h_3 = 0 \ , \qquad j_1(dw_s)^2 + j_2 dw_s + j_3 = 0 \ , \tag{53}$$

where

$$j_1 = J_1(h_2 / h_1)^2 - J_2(h_2 / h_1) + J_3 \ , \quad j_3 = J_1(h_3 / h_1)^2 - J_4 h_3 / h_1 + J_6 \ , \tag{54}$$

$$j_2 = 2J_1(h_2 h_3 / h_1^2) - J_2 h_3 / h_1 - J_4 h_2 / h_1 + J_5 \ , \ h_1 = d\tilde{a}_{2j} \ , \quad h_2 = d\tilde{a}_{3j} \ , \tag{55}$$

$$h_3 = d\tilde{a}_{1j} \ , \quad J_1 = d\tilde{\mathbf{a}}_2^{*T} d\tilde{\mathbf{a}}_2^* \ , \quad J_2 = 2d\tilde{\mathbf{a}}_2^{*T} d\tilde{\mathbf{a}}_3^* \ , \quad J_3 = d\tilde{\mathbf{a}}_3^{*T} d\tilde{\mathbf{a}}_3^* \ , \tag{56}$$

$$J_4 = 2(\Delta\tilde{\mathbf{a}}^{*i} + d\tilde{\mathbf{a}}_1^*)^T d\tilde{\mathbf{a}}_2^* \ , \qquad J_5 = 2(\Delta\tilde{\mathbf{a}}^{*i} + d\tilde{\mathbf{a}}_1^*)^T d\tilde{\mathbf{a}}_3^* \ , \tag{57}$$

$$J_6 = (\Delta\tilde{\mathbf{a}}^{*i} + d\tilde{\mathbf{a}}_1^*)^T (\Delta\tilde{\mathbf{a}}^{*i} + d\tilde{\mathbf{a}}_1^*) - (\Delta s)^2 \ . \tag{58}$$

The superscript star indicates that the quantity with it is divided by the scaling factors as in Eqn (45)

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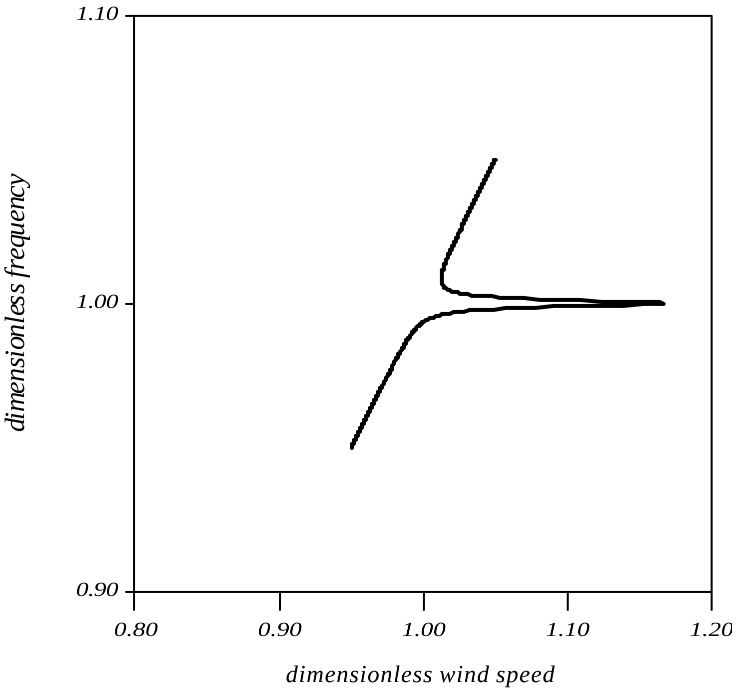

Fig.1 The non-dimensional response frequency

The increments $\boldsymbol{dw}$ and $\boldsymbol{dw}_s$ which give a positive value of $(\Delta\tilde{\mathbf{a}}^{*i+1})^T\Delta\tilde{\mathbf{a}}^{*i}$ are taken as the correct solution of Eqn (53). The iterations are repeated until the appropriately accurate solution is found.

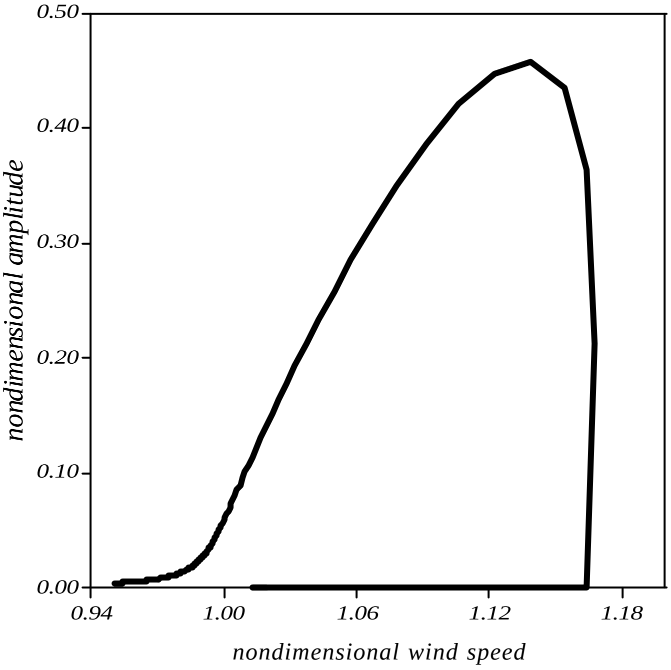

Fig.2 The non-dimensional amplitude of vibration

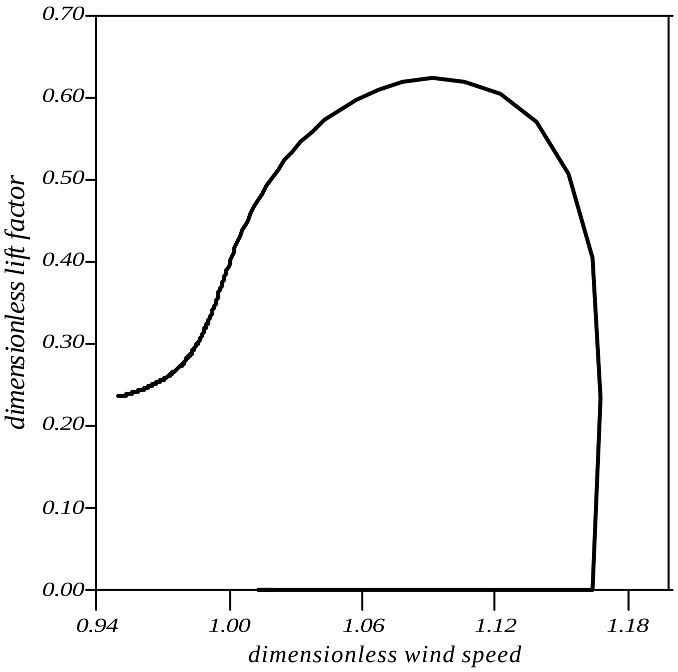

Fig.3 The non-dimensional amplitude of lift factor

6. NUMERICAL EXAMPLE

The numerical results of analysis of typical cylinder used in experiments are presented in Figs 1-3. The considered cylinder is elastically supported at both ends. In Fig.1, the non-dimensional frequency w/w_s versus non-dimensional wind speed defined by $2pSU/(Dw_s)$ is shown. The region where the lock-in phenomena occurs (i.e. where $w \approx w_s$) is clearly demonstrated. In Fig. 2, the non-dimensional amplitude of vibration w/D versus non-dimensional wind velocity is presented. The periodic vibrations with large amplitudes are observed in the synchronisation range. The amplitude of lift factor also versus the non-dimensional wind velocity is presented in Fig.3. These results shown that the peak of lift factor amplitude occurs at a lower wind speed than does the peak of cylinder amplitude. It is in agreement with experimental results.

7. FINAL REMARKS

In this paper, the computational formulation for non-linear vortex-induced vibration of beams is described. The presented method is an analytical-numerical one and, due to properties of the method, it is very efficient for the numerical study of system behaviour.

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