Non-local continuum mechanics based on fractional calculus

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Abstract

The new concept of non-local continuum body definition utilising the fractional calculus in under consideration. Non-locality appears as a consequence of the fractional derivative definition which bases on an interval contrary to the classical one whose definition is given in a point. The classical formulation is recovered as a special case of introduced generalisation.

Keywords: non-local models, fractional calculus, continuum mechanics

1. Introduction

Non-local formulations play essential role in the description of the material deformation and should be understood as an introduction of additional measures into the classical continuum mechanics approach (where all measures are defined in a point [16, 8]). Such addition implies that the information in the material point (e.g. strains level) contains somehow information from its finite surrounding what is in agreement with experimental observations. The size of this surrounding depends on a particular material (e.g. steel, ceramics, concrete) and is often called characteristic length scale of the material. It is now well established in mechanics that to cover some phenomena like e.g. scale effects or strain softening (due to lost of well-posedness of initial boundary value problems c.f. [10]) non-local formulations are necessary. It is then desirable to discover new concepts of non-local formulations in continuum mechanics.

First articles in this area were released in 1960s predicting phenomena such as stress concentration at holes, crack-tip stresses, bending stiffness of thin beams or stresses at free surfaces (cf. [6] and cited therein). Most recently non-local formulations are mainly applied to correctly predict softening range of material behaviour [14, 15, 18]. Independently of application, we single out two common ways to classify the introduction of the characteristic length scale, i.e. explicit [3, 2, 17] (e.g. via classical strain gradients) or implicit [12, 11, 5] (i.e. via relaxation time in Perzyna’s type viscoplasticity). All of them tend the modelling become more realistic according to experimental observations. In this paper we present the new concept of non-local continuum body definition utilising the fractional calculus.

2. Fractional continua

The hole idea bases of the fractional deformation gradients concept [13].

So, the regular motion of the material body $B$ can be written as

$$\mathbf{x} = \phi(\mathbf{X}, t),$$

while its inverse is

$$\mathbf{X} = \varphi(\mathbf{x}, t),$$

thus $\phi : B \rightarrow S$ is a $C^1$ actual configuration of $B$ in $S$, at time $t$. Now the generalisation of the classical deformation gradient and its inverse, utilising fractional calculus is as follows

$$\tilde{F}_X(\mathbf{X}, t) = \ell_X^{-\alpha} D^\alpha \phi(\mathbf{X}, t),$$

and

$$\tilde{F}_x(\mathbf{x}, t) = \ell_x^{-\alpha} D^\alpha \varphi(\mathbf{x}, t),$$

where $D^\alpha$ is a fractional differential operator in the sense of Riesz-Caputo (RC) [1], $\ell_X$ and $\ell_x$ are length scales in $B$ and $S$, respectively. We assume additionally that $\ell = \ell_X = \ell_x$.

Now we can introduce fractional counterparts of material and spatial line elements, namely

$$d\mathbf{x} = \tilde{F}_X d\mathbf{X},$$

and

$$d\mathbf{X} = \tilde{F}_x d\mathbf{x}.$$

In Figure 1 we present possible relations between the classical material and spatial line elements $(d\mathbf{X}, d\mathbf{x})$ with their fractional counterparts.

Figure 1: The relations between material and spatial line elements with their fractional counterparts $\tilde{F}_X = \tilde{F}_x \tilde{F}^{-1}_x$, $\tilde{F}_x = \tilde{F}_X \tilde{F}^{-1}_X$.
Based on fractional deformation gradient $\hat{F}_X$ or $\hat{F}_x$ or $\hat{F}_\alpha$ one can generalise all classical measures like Green-Lagrange strain tensor, Euler-Almansi strain tensor, right/left stretch tensors etc. - of course they are all non-local due to non-locality of fractional deformation gradient applied.

As mentioned for $\alpha = 1$ RC fractional derivative becomes classical one and length scale $\ell^{\alpha-1} = \ell^{1-1} = \ell^0 = 1$ do not influence the results, so we have:

$$F = \hat{F}_X \overset{\alpha}{=} \hat{F}_x^{-1} = \hat{F}_\alpha,$$

(7)

$$F^{-1} = \hat{F}_x^{-1} = \hat{F}_X = \hat{F}_\alpha^{-1},$$

(8)

$$\overset{\alpha}{F}_X = I, \quad \overset{\alpha}{F}_x = I,$$

(9)

dx = d\bar{x}, \quad dX = d\bar{X}.

(10)

3. Examples

The solution of the problem of elastic and thermoelastic deformation of fractional continua is considered. The governing equations of boundary value problems are defined and solved using generalised finite difference method [9, 4, 7]. The influence of applied length scale $\ell$ and the order of the fractional continua $\alpha$ is of special attention - c.f. Fig. 2 where the results of one dimensional tension under body loads is presented.

Figure 2: The comparison of displacements through the length $l$ of the body for length scale $\ell = 0.1l$ and order of fractional continua $\alpha \in \{0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$.

4. Conclusions

In the paper the generalisation of the classical continuum mechanics utilising fractional calculus is presented. The obtained description is non-local. The specific features of the formulation (the influence of length scale and order of fractional continua) are illustrated by analysis of the solution of different boundary value problems.

References


